Design of formal control laws for time-constrained partially observable discrete event systems with the presence of disturbances

I. Amama, S. Bouazza, S. Amari, H. Hassine, R. Chaari and M. Haddar.

***Abstract*— Real-world applications, including supply chains, flexible industrial systems, networked control, and urban transport, frequently exhibit the structure of Discrete Event Systems (DES). This study focuses on controlling DES subject to disturbances while ensuring strict time constraints, specifically addressing partially controllable and observable timed event graphs. The behavior of these graphs is modeled using Min-Plus algebra and linear inequalities, which are then used to formalize the constraints to be satisfied. An algebraic approach for designing control laws to meet these restrictions is proposed. Sufficient conditions for the existence of causal state feedback that guarantees specification satisfaction are established. The approach is validated through a disturbed supply chain application, where controllers are implemented as timed and marked places, acting as supervisors to prevent constraint violations.**

Index Terms— Discrete event systems, Min-Plus algebra, Temporal constraints, Timed event graphs, Control laws.

I. INTRODUCTION

Supply chains and manufacturing systems play a fundamental role in ensuring the efficient flow of materials, information, and goods from suppliers to end consumers [1]. Their optimization is essential for maintaining productivity, reducing costs, and minimizing disruptions caused by unforeseen events. Effective management of these systems enhances overall performance, ensuring timely deliveries and resource availability while adapting to operational constraints. The discrete evolution of these systems is particularly relevant in industries such as flexible assembly and transportation [2], where time constraints— including validity, strict duration, and time intervals—are critical. Neglecting these constraints may lead to forbidden states. In recent years, numerous studies have explored the control problem of DESs under various constraints and limitations. Given the prevalence of these limitations, addressing them is crucial for developing control laws that adhere to time constraints. Timed Event Graphs (TEGs), a specific class of Petri nets, have been widely studied due to their practical representation using dioid algebra and linear state equations. Linear inequalities are used to model the constraints that must be satisfied, as discussed in [3,4]. When solving feedback control problems for TEGs under strict time constraints using formal methods, researchers in [5,6] proposed a novel approach incorporating additional tools. The authors in [7] found that control laws may exist if there is an empty path between the control transition and the upstream transition of the constraint place. However, this condition—when no tokens are present—compromises system liveness. The approach in [5] addressed time constraints on empty paths in TEGs, in contrast to prior studies [4,8], which relaxed assumptions only for specific constrained places. In [9], an algebraic approach was proposed for computing control laws that ensure compliance with generalized marking constraints, along with causal feedback to enforce these limitations. Additionally, [10] introduced a mathematical method for synthesizing control laws that respect generalized mutual exclusion constraints using Min-Plus formalisms. The authors in [11] developed a control synthesis procedure for Networked Conflicting TEGs (NCTEGs) to meet upper temporal bounds on critical paths, addressing both fully and partially observable NCTEGs via Max-Plus algebra. In [12], an algebraic modeling framework using switching Max-Plus linear equations was proposed to characterize NCTEG behavior, though these studies did not account for uncontrollable transitions. Recent research has expanded the scope of TEG systems to address disruptions using idempotent semirings while ensuring additional criteria such as just-in-time constraints [14,15], optimal control updates to mitigate disturbances [13], and disassembly challenges [16,17]. For instance, [14,15] proposed an observer-based control system to solve the disturbance decoupling problem while maintaining just-in-time constraints using Max-Plus algebra. Meanwhile, [13] focused on optimally updating control inputs to minimize disturbance impacts while preserving just-in-time performance. Additionally, [16,17] addressed disruptions in TEG disassembly problems with marking constraints using Min-Plus algebra—[16] developed a control approach for partially observable systems, while [17] focused on fully observable systems, presenting analytical control methods. Further extending this research, [18] introduced a control method to enforce strict timing constraints in fully observable TEGs using Min-Plus algebra and state feedback control.

Building on this body of work, this paper investigates the control of strict temporal constraints in partially observable TEGs with disturbance transitions. Our focus is on regulating time limits on empty paths, where disturbances may occur and

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not all system states are directly observable. To ensure compliance with these constraints, we propose a control methodology based on Min-Plus equations to describe system behavior and inequalities to express temporal constraints. The control law is computed after validating necessary conditions, enabling disturbed systems to maintain strict timing requirements. To illustrate our approach, we apply it to a supply chain example subject to disturbances. Unlike existing methods, our approach effectively manages time constraints without suffering from combinatorial state explosion.

The remainder of this paper is structured as follows. Section 2 provides background on the Min-Plus model, TEGs with disturbance transitions, and temporal constraints. Section 3 presents the control design strategy, first addressing a single timing constraint and then extending it to multiple constraints. Section demonstrates the approach through a case study. Finally, Section 5 concludes the study and outlines future research directions.

II. Preliminaries and Basic Definitions

A. Timed event graphs with disturbance transitions

TEGs, a subclass of Petri nets, are characterized by each place having a single input and output transition, with all arcs weighted at 1. Temporal constraints apply to places, representing token retention time, and to transitions, representing franchising duration. Our analysis considers disturbed TEGs with two types of transition sources: t_u: The only controlled transition, whose timing function is determined using Min-Plus algebra; t_w: Uncontrollable transitions that cannot be influenced; t_i : Other transitions are classified as internal.

Uncontrollable input transitions that negatively impact system performance are termed disturbances. In supply chains, disruptions can originate from internal or external sources. Internal disturbances These arise from within the system and include: Unforeseen production issues, such as equipment failures [16,17],extended task durations, raw material shortages [18],unplanned operator absences. External disturbances These affect system inputs and outputs and often result from: Supplier delays, stock-outs due to timing mismatches in raw material availability [18], logistical disruptions (e.g., traffic congestion, transportation delays), frequent last-minute order changes by clients. Additionally, large-scale disruptions—such as health crises and pandemics (e.g., the COVID-19 outbreak [19])—can introduce severe and unpredictable disturbances into supply chain operations. In this study, t_w represents unpredictable or disturbed transitions. The sets T_y and T_β contain observable and unobservable transitions, respectively, with $T_y + T_\beta = N$. A path linking $t_i \in T_\beta$ to $t_j \in T_y$ is denoted by δ, with τ_δ and m_δ representing its timing and marking. A path that loops back to its starting node is called a Petri net loop.

B. Linear Min-Plus model

In the modeling of DES, key phenomena including synchronization, delays, and parallelism can be effectively described using Min-Plus algebra (also known as the Min-Plus dioid). This algebraic framework provides a linear representation for certain classes of systems that would otherwise require differential equations in conventional algebra. So, we study the Min-Plus algebra, denoted by $\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ .The min application is shown as the $\oplus$,$\forall\, \sigma, \omega \epsilon\, \overline{\mathbb{R}}_{min} : \sigma \oplus \omega = min(\sigma, \omega)$, or ε=+∞ the neutral component. "Classical addition" refers to the second law ⊗", that is $\forall\, \sigma, \omega \epsilon\, \overline{\mathbb{R}}_{min} : \sigma \otimes \omega = \sigma + \omega$ where e=0 denoted the neutral component. The Min-Plus matrix algebra element that represents the intersection of a matrix A is i^{th} row and j^{th} column is called A_{ij}. The set $n * m$ matrices over $\overline{\mathbb{R}}_{min}$ for $n, m \epsilon \overline{\mathbb{R}}_{min}^{n*m}$ is given by $\overline{\mathbb{R}}_{\min}^{n*m}$ and the next operations, where $\forall A, B \epsilon \overline{\mathbb{R}}_{min}^{n*m}, \forall C \epsilon \overline{\mathbb{R}}_{min}^{m*p}$ are determined by:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, \forall\, i = 1, \dots, n\, \forall\, j = 1, \dots, m.$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{m} A_{ik} \otimes C_{kj}, \forall\, i = 1, \dots, n\, \forall\, j = 1, \dots, p.$$

Lemma. [3] The implicit equation $x = A \otimes x \oplus B$ defined over a complete dioid D admits $x = (A^* \otimes B)$ as a solution, with A^* is the Kleene star of matrix A, such that:

$$\forall A \in D, A^* = \bigoplus_{i \geq 0} A^{\otimes i}$$

Where, $(A^{\otimes i} = A^{\otimes i-1} \otimes A)$ and $A^0 = I$, note that I is the unit matrix, with entries equal to e on the diagonal and ε elsewhere.

C. Path under strict temporal constraints

Modern supply chain operations typically involve strict temporal constraints that are often localized to specific process paths. To maintain critical operational requirements, these systems demand additional control measures. This necessitates the development of efficient control strategies using the path-constrained formulation of Min-Plus algebra. This research focuses on controlling disrupted TEGs under rigorous temporal constraints. Beyond regulating token residence times, our approach specifically addresses path-based constraints within disturbed TEG systems. The primary objective is to derive control laws that: Ensure strict adherence to task-specific time limitations, prevent violations of temporal specifications, maintain system performance despite disruptions These temporal restrictions may also apply to the paths that constitute the disrupted TEG. For a strictly limited path ξ, the upstream and downstream transitions t_j and t_i respectively, and the associated time interval $[\tau_{ij}^{min_\xi}, \tau_{ij}^{max_\xi}]$. The temporal constraint along the path from t_j to t_i is described by the following inequality:

$$x_i^\xi(t) \geq m_{ij}^\xi \otimes x_j^\xi\left(t - \tau_{ij}^{max_\xi}\right) \quad (1)$$

where m_{ij}^ξ is the sum of all places initially designated as part of the path ξ_{ij}. The maximum upper bound, given by $\tau_{ij}^{max_\xi}$, represents the sum of all delays associated with the places along the path ξ and is available for $\overline{\mathbb{R}}_{min}$.

III. Control Design Strategy

A. Formalization of the control problem statement

The temporal dynamics of a TEG can be formally represented as a directed graph whose behavior is captured through dynamic equations in the time domain. Specifically, these equations are designed to count the number of transitions that have occurred up to a given time t. This behavior is captured by linear equations utilizing counter functions $\theta_i(t)$ associated with each transition ti in the graph. The uncontrollable transitions t_w disrupt the system's operation. The function $x_i(t)$ defines the firing count of the transition t_i at time T. The function $u(t)$ counts the controlled transition sources, while the counter function $w(t-\tau)$counts the perturbed transitions at a given instant. The dynamic behavior of this graph is described as follows:

$$\theta(t) = \oplus_{\tau=0}^{\tau max} (A_\tau \otimes \theta(t-\tau) \oplus B_\tau \otimes u(t-\tau) \oplus Q_\tau \otimes w(t-\tau) \quad (2)$$

where, $A_\tau \in \underline{R}_{min}^{n*n}$ is the matrix where the entry $A_{\tau,ij}$ equals the quantity of initial tokens m_{ij} at a place p_{ij}. The corresponding time is used if a place is present, and ε otherwise. The matrix $B_\tau \in \underline{R}_{min}^{n*m}$ represents the number of direct initial tokens for controllable transitions. The matrix $Q_\tau \in \underline{R}_{min}^{n*m}$ represents the number of direct initial tokens for uncontrollable transitions. To transform Equation (2) into the state form of the system's dynamic behavior, we first decompose each place with a time delay τ>1 into multiple places, each with a unit time delay of 1, and introduce visible intermediate transitions. Each of these intermediate transitions is associated with a counter. Once this decomposition is complete, and assuming all delays in the perturbed TEG are uniform, the behavior can be represented in state space form. In this scenario, (t) is extended to the state space vector x(t). Consequently, the behavior of the extended graph is described by the following explicit equation:

$$x(t) = A \otimes x(t-1) \oplus B \otimes u(t) \oplus Q \otimes w(t) \quad (3)$$

where A=$A_0^* \otimes A_1$, B=$A_0^* \otimes B$, Q=$A_0^* \otimes Q$

All path markers connecting internal transitions t_i with zero delay are represented by A_0. This matrix contains the markers for the non-delayed pathways joining the internal transitions. Similarly, A_1 represents the same pathways but with a delay of 1. Additionally, B_0 and Q_0 denote the markers for zero-delay pathways between the control transitions t_u and the internal transitions t_i, and between the internal transitions t_i and the disturbed transitions t_w, respectively. For every integer $\boldsymbol{\tau}$ such that $\boldsymbol{\tau} \geq 1$, the following form can be used for the state representation (3):

$$x(t) = A^\tau \otimes x(t-\tau) \oplus [\oplus_{k=0}^{\tau-1} A^K \otimes B \otimes u(t-k)] \oplus \left[\oplus_{g=0}^{\tau-1} A^g \otimes Q \otimes w(t-g)\right] \quad (4)$$

Let us assume that τ = ϕ in (4). Then $x_i^\xi(t)$ can be represented by the following expression:

$$x_i^\xi(t) = \oplus_{r=1}^{N} (A^\phi)_{ir}^\xi \otimes x_r(t-\phi) \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes B)_i^\xi \otimes u(t-k)] \oplus [\oplus_{g=0}^{\phi-1} (A^g \otimes Q)_i^\xi \otimes w(t-g)] \quad (5)$$

Property. In a TEG with disturbance inputs, every loop contains at least one observable transition. Moreover, for each unobservable transition $t_i \epsilon\ t_\beta$, there exists a path δ leading to an observable transition $t_i\ \epsilon\ t_y$. This relationship allows the counter associated with an unobservable transition t_i to be expressed as a function of the observable transition t_j.

$$x_i^\xi(t) \leq m_\delta \otimes x_j^\xi(t-\tau_\delta) \quad (6)$$

where m_δ and τ_δ represent the number of tokens and the timing along path δ, respectively.

Definition 1. Observability [20]: A transition t_i is considered observable if it is directly connected to an output transition. A TEG is said to be totally observable if all its internal transitions can be observed.

Definition 2. Partially observable TEG [20]: A TEG is partially observable if it contains at least one unobservable transition $t_i \in t_\beta$, where T_β represents the set of unobservable transitions.

B. Control law design approach

a) *Disturbed TEG with one constraint*

We analyze and apply a TEG with disturbances under time restriction for a path. The state equation for this system is defined by Equation (3), while the constraint is given by Inequality (1). The values τ_α and m_α represent the cumulative delay and the total number of markers along the path, respectively. The terms t_u and t_j^ξ denote the counter functions u(t) and $x_j^\xi(t)$, respectively. The relationship between the counter functions u (t) and $x_j^\xi(t)$ is expressed by the following inequality:

$$x_j^\xi(t) \leq m_\alpha \otimes u(t-\tau_\alpha)$$

Using the definitions of t_u and t_j^ξ, along with $\boldsymbol{\tau}=\tau_\alpha$ the following can be derived :

$$x_j^\xi(t) \leq (A^{\tau_\alpha} \otimes B)_{uj} \otimes x_u(t-\tau_\alpha)$$

Considering that $x_u(t) \leq$ u(t), it is evident that :

$$x_j^\xi(t) \leq (A^{\tau_\alpha} \otimes B)_{uj} \otimes u(t-\tau_\alpha) \quad (7)$$

The explicit equation that follows can be obtained by reapplying (4) with $\phi = \tau$.

$$x(t) = \oplus_{r=1}^{N} \left(A^\phi\right)_{ir} \otimes x_r(t-\phi) \oplus [\oplus_{k=0}^{\phi-1} (A^K \otimes B)_{i\xi} \otimes u(t-k) \oplus \left[\oplus_{g=1}^{\phi-1} (A^g \otimes Q)_{i\xi} \otimes w(t-g)\right]$$

for $K = 1\ to\ \phi - 1\ and\ g = 1\ to\ \phi - 1$. All integers from r=1 to N can access it; this is the key to getting the next result.

Theorem 1. We consider a partially observable TEG with disturbance transitions and a timed constraint of the form (1). The control law is characterized by the following formula:

$$u(t) = F_1' \otimes x_r(t-1) \oplus F_1'' \otimes x(t-\tau_\delta-1) \oplus F_2 \otimes w_h(t-1) \quad (8)$$

with

$$F_1' = \oplus_{r\epsilon y} \left((A^\phi)_{ir}^\xi - (A^{\tau_\alpha} \otimes B)_{uj}^\xi - m_{ij}^\xi\right),$$

$$F_1'' = \oplus_{\zeta=1}^{\beta} \oplus_{r\epsilon y} \left(m_\delta - (A^{\tau_\alpha} \otimes B)_{uj}^\xi - m_{ij}^\xi\right) \otimes (A^\phi)_{ir}^\xi,$$

$$F_2 = \oplus_{h=1}^{H} \oplus_{h\epsilon y} ((A^g \otimes Q)_i^\xi - (A^{\tau_\alpha} \otimes B)_{uj}^\xi - m_{ij}^\xi),$$

This ensures that constraint (1) is satisfied, provided the following conditions are met:

$$(A^k \otimes B)_i^\xi \geq (A^{\tau_\alpha} \otimes B)_{uj}^\xi + m_{ij}^\xi, \quad \text{for } k = 0 \text{ to } \phi\text{-}1 \quad (9)$$

$$(A^g \otimes Q)_i^\xi \geq (A^{\tau_\alpha} \otimes B)_{uj}^\xi + m_{ij}^\xi, \quad \text{for } g = 0 \text{ to } \phi\text{-}1 \quad (10)$$

Proof 1. By combining equations (7) and (5), the timed constraint in equation (1) is explicitly satisfied for a TEG with disturbance inputs, where ξ represents the path constrained.Thus, satisfying the time constraint (1) requires meeting the following three conditions: (11), (12), and (13).

$$\oplus_{r=1}^{N}\left((A^{\phi})_{ir}^{\xi}-m_{ij}^{\xi}\right)\otimes x_r\left(t-\phi+\tau_{ij}^{max_\xi}\right)\geq (A^{\tau_\alpha}\otimes B)_{uj}^{\xi}\otimes u(t-\tau_\alpha) \quad (11)$$

$$\oplus_{k=0}^{\phi-1}((A^{K}\otimes B)_{i}^{\xi}-m_{ij}^{\xi})\otimes u(t-k+\tau_{ij}^{max_\xi})\geq (A^{\tau_\alpha}\otimes B)_{uj}^{\xi}\otimes u(t-\tau_\alpha) \quad (12)$$

$$\oplus_{g=1}^{\phi-1}((A^{g}\otimes Q)_{i}^{\xi}-m_{ij}^{\xi})\otimes w(t-g+\tau_{ij}^{max_\xi})\geq (A^{\tau_\alpha}\otimes B)_{uj}^{\xi}\otimes u(t-\tau_\alpha) \quad (13)$$

Our goal is to develop a realizable controller based on $x(t-1)$ and $w(t-1)$. Given that $\phi=\tau_{ij}^{max_\xi}+\tau_\alpha+1$ and considering $u(t)\geq u(t-1)$,, inequalities (14) and (15) can be rewritten as follows:

$$(A^{k}\otimes B)_{i}^{\xi}\geq (A^{\tau_\alpha}\otimes B)_{uj}^{\xi}+m_{ij}^{\xi},\ \text{for k=0 to } \phi-1 \quad (14)$$

$$(A^{g}\otimes Q)_{i}^{\xi}\geq (A^{\tau_\alpha}\otimes B)_{uj}^{\xi}+m_{ij}^{\xi},\ \text{for g=0 to } \phi-1 \quad (15)$$

This inequality defines the requirement of Theorem 1. If conditions (11) and (13) hold, the control law ensuring time constraint (1) follows from inequalities (14) and (15). A feasible controller for a partially observable TEG with disturbance transitions is given by:

$$u(t)=[(\oplus_{r=1}^{N}(A^{\phi})_{ir}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes x(t-1)] \oplus[\oplus_{h=1}^{H}(A^{g}\otimes Q)_{ih}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes w_h(t-1)] \quad (16)$$

Since its equation depends on the internal transitions of the TEG, including unobservable ones, this controller is not directly practical. However, using Property 1, it can be transformed into a realizable controller, as every loop in the disturbed TEG contains at least one observable transition. To achieve this, as shown in inequality (17), the controller formula is divided into two parts: one for observable transitions and another for unobservable ones.

$$u(t)\leq[(\oplus_{r=1}^{N}(A^{\phi})_{ir}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes x_r(t-1)]\oplus[\oplus_{r\epsilon\beta}\left((m_\delta-(A^{\tau_\alpha}B)_{uj}^{\xi}-m_{ij}^{\xi})+(A^{\phi})_{ir}^{\xi}\right)\otimes x_r(t-1)]\oplus[\oplus_{h=1}^{H}\oplus_{g=0}^{\phi-1}(A^{g}\otimes Q)_{ih}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes w_h(t-1)] \quad (17)$$

Based on Property 1, any counter function of an unobservable transition can be expressed in terms of an observable transition. Thus, the control law in terms of observed transitions is given by the following equation.

$$u(t)=[(\oplus_{r\epsilon y}(A^{\phi})_{ir}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes x(t-1)]\oplus[(\oplus_{\zeta=1}^{\beta}(m_\delta-(A^{\tau_\alpha}B)_{uj}^{\xi}-m_{ij}^{\xi})+(\oplus_{r\epsilon y}(A^{\phi})_{ir}^{\xi})\otimes x(t-\tau_{\delta_\zeta}-1)]\oplus[\oplus_{g=1}^{\phi-1}\oplus_{h\epsilon y}(A^{g}\otimes Q)_{ih}^{\xi}-(A^{\tau_\alpha}\otimes B)_{uj}^{\xi}-m_{ij}^{\xi})\otimes w_h(t-1)]$$

Remark 1. Negative markings in controlled places (i.e., non-practicable places) indicate a non-causal control law when the sufficient condition for its existence is not met. This highlights the need for further study on necessary and sufficient conditions for control laws. If the control law is non-causal, a feasible one can be obtained by replacing e for each negative term in the coefficients F_1', F_1'', and F_2 of Theorem 1.

$$u(t)=[\oplus_{r\epsilon y}e\otimes x_r(t-1)]\oplus[\oplus_{\zeta=1}^{\beta}e\otimes x_r(t-\tau_{\delta_\zeta}-1)]\oplus[\oplus_{h=1}^{H}\oplus_{h\epsilon y}e\otimes w_h(t-1)]$$

b) Disturbed TEG with multi constraints

This subsection addresses the control synthesis of a TEG with disturbance transitions and a one source transition. For v=1 to V, the symbols ξ_v represent the V constrained paths. Let m_v^{ξ}, τ_v^{ξ}, and $\tau_v^{max\xi}$ denote the initial marking, minimal delays, and maximal delays, respectively, for each constrained path ξ. Additionally, let t_d^{ξ} and $t_{d\prime}^{\xi}$ represent the input and output transitions of path v, respectively. The functions x_d^{ξ} and $x_{d\prime}^{\xi}$ denote the counters for these transitions. Moreover, λ_v^{ξ} represents the cumulative delay along the path from t_u to t_j^{ξ}. The following inequalities represent these limitations:

$$x_{d\prime}^{\xi}(t)\geq m_v^{\xi}\otimes x_d^{\xi}(t-\tau_v^{max_\xi})\quad \text{for } v=1 \text{ to } V \quad (18)$$

As indicated by Inequality (8), the control law (t) is the one that satisfies each corresponding constraint on the paths.

Theorem 2. The control law for a TEG with disturbance transitions, derived from equation (3) and subject to multiple timed constraints of form (18), is formulated as follows:

$$u(t)=\oplus_{v=1}^{V}u_v(t) \quad (19)$$

$$u(t)=\oplus_{v=1}^{V}[(\oplus_{r\epsilon y}F_{1v}'\otimes x_r(t-1))\oplus(\oplus_{\zeta=1}^{\beta}\oplus_{r\epsilon y}F_{1v}''\otimes x_r(t-\tau_{\delta_\zeta}-1))\oplus(\oplus_{h=1}^{H}F_{2v}\otimes w_h(t-1))]$$

where

$$F_{1v}'=\oplus_{r\epsilon y}((A^{\phi})_{d'r}^{\xi}-(A^{\tau_\alpha}\otimes B)_{ud}^{\xi}-m_v^{\xi}),$$

$$F_{1v}''=\oplus_{\zeta=1}^{\beta}\oplus_{r\epsilon y}\left((m_{\delta_\zeta}-(A^{\tau_\alpha}\otimes B)_{ud}^{\xi}-m_v^{\xi})+(A^{\phi})_{d'r}^{\xi}\right),$$

$$F_{2v}=\oplus_{h=1}^{H}\oplus_{h\epsilon y}\left((A^{g}\otimes Q)_{d}^{\xi}-(A^{\tau_\alpha}\otimes B)_{ud}^{\xi}-m_v^{\xi}\right),$$

As long as the following conditions are fulfilled, constraint (18) is satisfied.

$$(A^{k}\otimes B)_{d'}^{\xi}\geq (A^{\tau_\alpha}\otimes B)_{ud}^{\xi}+m_v^{\xi},\quad \text{for } v=1 \text{ to } V \quad (20)$$

$$(A^{g}\otimes Q)_{d'}^{\xi}\geq (A^{\tau_\alpha}\otimes B)_{ud}^{\xi}+m_v^{\xi},\quad \text{for } v=1 \text{ to } V \quad (21)$$

Proof 2. If conditions (9) and (10) hold, the controller u(t), as defined in Theorem 1, ensures compliance with time constraint (1). Additionally, if condition (18) is met, the control law $u_v(t)$ prevents the violation of the V^{th} constraint among timed constraints. The control laws ensuring the fulfillment of all V constraints (18) are given by:

$$u(t) = \bigoplus_{v=1}^{V} u_v(t) \qquad \square$$

IV. APPLICATION

The supply chain model (Figure 1) represents an integrated manufacturing and distribution network with the following key components: The manufacturing system includes detailed production steps, such as the assembly of raw materials, the use of robotic packaging systems, automated handling via mechanical arms, and conveyor-based transportation to company warehouses. Once the finished products are stored and ready for shipment, they are transported to two distinct distribution centers via trucks. These centers are equipped to efficiently manage inventory and prepare shipments. From the distribution centers, the transportation network becomes multimodal, incorporating maritime shipping and air freight to support international distribution. The supply chain concludes with two separate warehouses, each serving different regions or markets. These warehouses act as final storage points before products are distributed to retailers or end customers. This representation highlights a modern and sophisticated supply chain, integrating automation, diversified transportation modes, and strategic distribution practices to maximize efficiency and responsiveness. We assume that only transitions t_1, t_2, t_4 , and t_5 are observable in this model.

TEG of this supply chain models the flow of goods from the availability of raw materials to their final delivery. The release of raw materials initiates the assembly process (t_1), which depends on the operator's capacity (p_{12}). Once assembled (p_{21}), the products are packaged by Robot 1 (t_2), requiring packaging materials (p_{23}). Then, Robot 2 transfers the packaged products (t_3, p_{43}) to Robot 3 (p_{54}), which directs them to the company's warehouse (t_4), depending on its capacity (p_{45}). The goods are then distributed through different transportation modes: Trucks 1 and 2 deliver them to distribution centers 1 and 2 (t_5, p_{65}, p_{75}), subject to their availability (p_{56}, p_{57}), with the possibility of delays or breakdowns (t_{5w}).

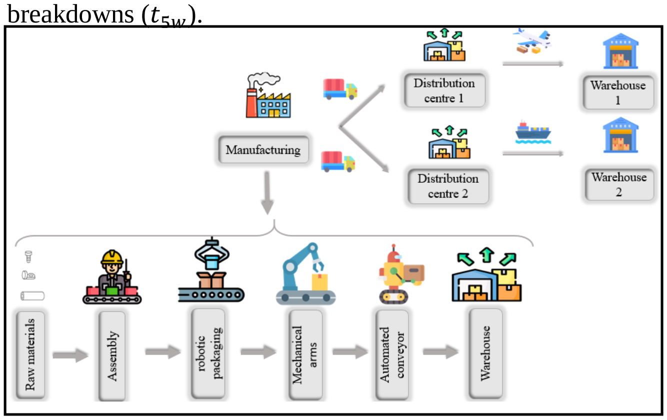

Figure 1. Description of the proposed supply chain

Meanwhile, part of the cargo is transported by air (t_6, p_{86}) or by sea (t_7, p_{97}), depending on the availability of these means (p_{68}, p_{79}). Finally, deliveries are made to the final depots (t_8 for p_{86} and t_9 for p_{97}), completing the logistics cycle. This model enables the analysis of time performance and the impact of disruptions on the supply chain. The TEG of this supply chain model includes one input transition, as shown in Figure 2, which represents the source transition t_u . Additionally, there is one disturbance transition t_w that impacts the critical time of the supply chain.

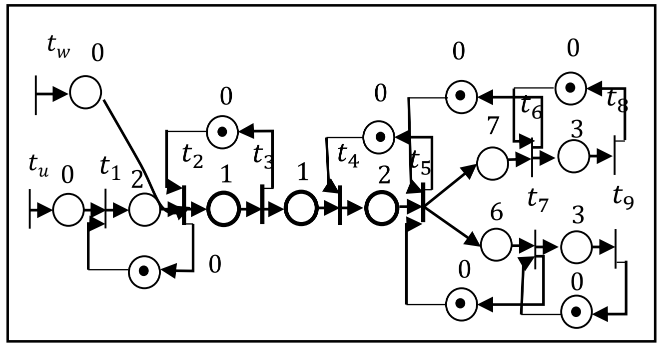

Figure 2. TEG of supply chain

To do that, we begin with the modeling stage and use equation: $\Theta(t) = A_0 \otimes \Theta_0(t) \oplus A_1 \otimes \Theta_1(t-1) \oplus A_2 \otimes \Theta_2(t-2) \oplus A_3 \otimes \Theta_3(t-3) \oplus A_6 \otimes \Theta_6(t-6) \oplus A_7 \otimes \Theta_7(t-7) \oplus B \otimes u(t) \oplus Q \otimes w(t)$.
with $\Theta_i = (\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6, \Theta_7, \Theta_8, \Theta_9)^t$ and $\varepsilon = +\infty$ to obtain the flowing matrices.
Before performing the calculations, we must first derive the system's state equation. To do so, We decompose the places with a timing delay strictly greater than 1 into several successive places, each with a unitary delay equal to 1.
The state space equation that results is:

$$x(t) = A \otimes x(t-1) \oplus B \otimes u(t) \oplus Q \otimes w(t)$$

With the path ξ_{52} running from t_2 to t_5. We attach an upper bound τ_{52}^{ξ}, which is $\tau_{52}^{max_\xi}$ time units. The marking along the path, represented by m_{52}^{ξ}, is likewise equal to $m_{52}^{max_\xi} = e$. The constrainted path is then defined by the inequality that follows:

$$x_5^{\xi}(t) \geq e \otimes x_2^{\xi}(t-4).$$

As an example, the total delay between t_2^{ξ} and t_u then $\tau_\alpha = 2$. Thus, we have $\phi = \tau_{ij}^{max} + \tau_\alpha + 1 = \tau_{52}^{max\xi} + \tau_\alpha + 1 = 7$ equation (4) takes on the following form:

$$x(t) = A^7 \otimes x(t-7) \oplus [\bigoplus_{k=0}^{6} A^k \otimes B \otimes u(t-k)] \oplus [\bigoplus_{g=0}^{6} A^g \otimes Q \otimes w(t-g)]$$

The initial marking is $m_\alpha = 0$. We have $(A^7)_{5r} = (1,2,2,\varepsilon,1,\varepsilon,\varepsilon,\varepsilon,\varepsilon,1,1,\varepsilon,2,\varepsilon,3,\ \varepsilon,2,2,\ \varepsilon,3,\varepsilon,2,\ \varepsilon,\varepsilon,\varepsilon,\varepsilon)$;
and $(A^k \otimes B)_{5,1}^{\xi} = [\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,e]$, for $k =$ 0 to 6, for $g = 0$ to 6, where $(A^2 \otimes B)_2^{\xi} = \mathrm{e}$, $(A^g \otimes Q)_{5,1}^{\xi} = [\varepsilon,\varepsilon,\varepsilon,\varepsilon,e,1,1]$. Then we verify the two conditions as follows:

$(A^k \otimes B)_{5,1}^{max_\xi} \geq (A^2 \otimes B)_{2,1}^{max_\xi} + m_{52}^{\xi}$, for k=0 to 6,

$(A^g \otimes Q)_{5,1}^{max_\xi} \geq (A^2 \otimes B)_{2,1}^{max_\xi} + m_{52}^{\xi}$, for g=0 to 6,

Following the verification of the adequate criteria, Theorem 4 provides the following initial control law, which guarantees that the path's constraint is respected:
$u(t) = 1 \otimes x_1(t-1) \oplus 2 \otimes x_2(t-1) \oplus 2 \otimes x_3(t-1) \oplus 2 \otimes x_5(t-1) \oplus 1 \otimes x_{10}(t-1) \oplus 1 \otimes x_{11}(t-1) \oplus 2 \otimes x_{13}(t-1) \oplus 3 \otimes x_{15}(t-1) \oplus 2 \otimes x_{17}(t-1) \oplus 2 \otimes x_{18}(t-1) \oplus 3 \otimes x_{20}(t-1) \oplus 2 \otimes x_{22}(t-$

$1) \oplus e \otimes w(t-1)$.

The control law guaranteeing is simplified to the next one $u(t) = 1 \otimes x_1(t-1) \oplus 2 \otimes x_2(t-1) \oplus 3 \otimes x_5(t-4) \oplus e \otimes w(t-1)$.

The proposed method utilizes TEG models and Min-Plus algebra to design control laws that ensure compliance with temporal constraints along specific TEG paths. These control laws are implemented via monitor places—modeled as marked and timed components—integrated into the original TEG structure (Figure 3). Represented as red double circles, these monitors act as supervisors, preventing constraint violations. The control input *u(t)* depends on four supervisory places: p_{uw} (unmarked, delay 1), $p_{\mathrm{u,1}}$ (1 token, delay 1), $p_{\mathrm{u,2}}$ (2 tokens, delay 1), and $p_{\mathrm{u,5}}$ (3 tokens, delay 4). These supervisory places enforce the temporal constraints through their structure and timing behavior. In addition, the implementation of these analytical control methods is simple and memory-efficient.

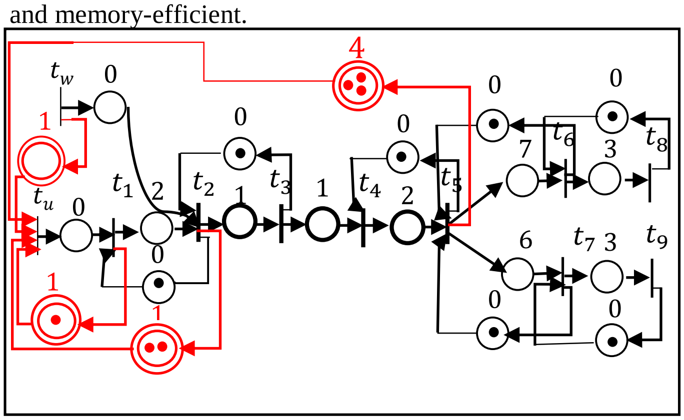

Figure 3. The controlled TEG for a considered supply chain

V. Conclusion

This paper presents a novel control design methodology for partially observable TEGs operating under disturbances, guaranteeing strict temporal constraint satisfaction. Our approach establishes sufficient conditions for control law synthesis that prevent constraint violations even with uncontrollable transitions, while relaxing observability requirements through path-based assumptions. The implementation framework computes feedback control using observable transition counters and enforces constraints through monitor places, with effectiveness demonstrated via a supply chain case study. While currently limited to a specific Petri net subclass and dependent on structural observability assumptions, the methodology opens several research directions: methodological enhancements for improved robustness and control optimization; theoretical extensions to broader Petri net classes and multi-input/multi-constraint systems; and practical applications adapting the framework to complex real-world scenarios and industrial control architectures.

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