On the Design of Interval Observers for Discrete-Time Linear Switched Systems without Using Similarity Transformations

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Abstract— **This paper presents synthesis methods of Interval Observers (IO) for discrete-time linear switched systems subject to additive unknown-but-bounded process and measurement noises. The novelty of the proposed methods consists in the designing of IO directly in the original state coordinates of the systems. This enables to: (i) mitigating the wrapping effect related to the classical use of similarity transformations; (ii) avoiding the impulsive behavior of the estimation error dynamics, mostly generated by the use of different similarity transformation for each mode of the switched system; (iii) reducing online computational effort. In addition, Bilinear Matrix Inequalities (BMI) and Linear Matrix Inequalities (LMI) conditions are established to check the existence and to compute stabilizing observer gains. The obtained theoretical results are supported by numerical simulations.**

I. INTRODUCTION

In many engineering areas, state estimation in the presence of different sources of uncertainty (modeling error, state disturbance, measurement noise, etc.) is a fundamental and challenging problem. Theoretically, based on a mathematical model of the system and its input and output data, an observer has to reconstruct the internal state of this system. Naturally, the accuracy and precision of the estimated state vector depends mainly on the quality of the used models to design observers and the accuracy of the available data. In the case of linear systems, under some statistical assumptions on the uncertain parts of the systems, numerous methods have been introduced in the literature to solve efficiently this problem. For example, classical Kalman filter [1] is applied to discrete-time systems while Luenberger observer [2], with an LQE gain tuning approach, is used to deal with continuous-time systems. However, in many real world applications, systems uncertainties are poorly-known and no probability density functions are available to describe them accurately. To rise above this problem, the concept of interval observer is proposed for biological systems [3] in order to estimate trajectory tubes containing, in a guaranteed way, the real state vector of the system. After this seminal work, the interval observer design problem has gained an increasing interest. Many methods have been developed to deal with different classes of continuous-time/discrete-time dynamical systems. For instance, the case of linear systems is studied in [4], [5], [6], the case of LTV/LPV systems is considered in [7], [8], while the case of nonlinear systems is tackled [9], [4], [10]. This concept has been also applied to some classes of linear switched systems in both continuous-time [11], [12], [13], [14] and discrete-time [15], [16], [17] framework. The proposed approaches rely mainly on the positive systems theory [18]. Their core idea consists in combining constant/time-varying similarity transformations with observer gain design procedures to ensure both stability and positivity of the observation error. By construction, the dynamics of this observation error inherits the switched behavior of the systems, which rends more challenging ensuring its stability without losing its positivity.

As pointed out in [20], although coordinate transformation methods have shown their effectiveness to ensure the sought positivity property of the estimation error, they affect negatively the estimation performance (accuracy and convergence rate). However, to deal with this issue alternative methods have been introduced in the literature for systems with single working mode. For instance, *Recursive design of input to state stable interval observers* for triangular nonlinear systems [21] and *Internal Positivity Decomposition* method for linear systems [6].

The goal of this work consists in providing convenient conditions that allow one to design interval observers for discrete-time switching linear systems in their original basis of the state variables, without requiring any similarity transformation. The proposed conditions are presented in terms of BMI and LMI constraints that can be solved by many available solvers in the literature.

The paper is organized as follows. First, preliminaries are presented in Section II. The structure of the proposed interval observer and its existence conditions are introduced in Section III. Then, design methods of the observer gains are given in Section IV. Simulation results are presented in Section V to show the merit of the proposed approach and to compare its performance with that provided by another method borrowed from the literature.

II. PRELIMINARIES

A. Notations

By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ one denotes the sets of non-negative integers, real numbers and non-negative real numbers, respectively. For a given real matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ is its induced matrix norm and one denotes by $\|A\|_\infty$ its infinity norm. For the sake of simplicity, any matrix in $\mathbb{R}^{p\times m}$ with all entries are zeros is denoted by 0 and by I_p one denotes an identity matrix of dimension $p\times p$. In what follows, when inequality operators are applied between vectors or matrices they should be understood element-wise. That is, for $A = (a_{i,j}) \in \mathbb{R}^{p\times m}$ and $B = (b_{i,j}) \in \mathbb{R}^{p\times m}$, $A \geq B$ implies that $a_{i,j} \geq b_{i,j}$ for all $i \in \{1,\dots,p\}$, $j \in \{1,\dots,m\}$. In addition, the max operator $C = \max\{A,B\}$ returns a real matrix $C = (c_{i,j})$ such that $c_{i,j} = \max\{a_{i,j}, b_{i,j}\}$. By $|A|$ one denotes the component-wise absolute value of a real matrix $A \in \mathbb{R}^{p\times m}$

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that can be computed as follows, $|A| = A^+ + A^-$ where $A^+ = \max\{A, 0\}$ and $A^- = A^+ - A$. A square real matrix $M \in \mathbb{R}^{n\times n}$ is an M-matrix if its off-diagonal entries are non-positive and its diagonal elements are non-negative. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative definite if $\upsilon^T P \upsilon < 0$ is true for all non-zero vectors $\upsilon \in \mathbb{R}^n$. This matrix is denoted by $P \prec 0$ and by $P \preceq 0$ one denotes a semi-negative definite matrix. The set of measurable and locally essentially bounded signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with bound L_∞-norm will be denoted by $\mathscr{L}_\infty$.

B. Definitions and Propositions

Definition 1 (Stieltjes matrix [22]): Any positive definite matrix is said a Stieltjes matrix if its off-diagonal elements are non-positive. Moreover, Stieltjes matrices are non-singular and their inverse matrices are symmetric non-negative matrices.

It is worth pointing out that, the converse of Definition 1 is not necessary true. In what follows, the set of Stieltjes matrices of dimension $n \times n$ is denoted by $\mathscr{S}^{n\times n}$.

Definition 2 (Schur complement [23]): Let Q and R two real square and symmetric matrices. If R is invertible, the following condition

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \preceq 0$$

is equivalent to $R \preceq 0$ and $Q - SR^{-1}S^\top \preceq 0$.

Definition 3 (Positive systems [18]): A discrete-time linear system described by $x(k+1) = Ax(k) + \phi(k)$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is positive if: (i) A is a nonnegative matrix; and (ii) $\phi(k)$ is a nonnegative vector.

Property 1: Any solution of a positive system, initialized at a nonnegative initial state vector $x(k_0) \geq 0$, stays nonnegative for all $k \geq k_0$. That is,

$$\forall x(k_0) \geq 0 \Longrightarrow x(k) \geq 0,\ \forall k \geq k_0 \tag{1}$$

Proposition 1: Let A_u and A_l be two nonnegative matrices in $\mathbb{R}^{n\times n}_{\geq 0}$ such that $A_u \geq A_l$, and $\phi_u(k)$ and $\phi_l(k)$ be two nonnegative vectors $\mathbb{R}^n_{\geq 0}$ such that $\phi_u(k) \geq \phi_l(k)$ for all $k \geq k_0$. Then, the solutions of the following positive systems

$$\begin{array}{rcl} x_u(k+1) & = & A_u x_u(k) + \phi_u(k) \\ x_l(k+1) & = & A_l x_l(k) + \phi_l(k) \end{array} \tag{2}$$

satisfy

$$x_u(k) \geq x_l(k),\ \forall k \geq k_0 \tag{3}$$

provided that $x_u(k_0) \geq x_l(k_0) \geq 0$.

Proof: The proof is immediate from the preserving order property of positive systems. ■

Proposition 2: Let A_P and A_N be two positive matrices in $\mathbb{R}^{n\times n}_{\geq 0}$, and define by $A = A_P - A_N$, Then one obtains:

$$A_P \geq A^+ \text{ and } A_N \geq A^-$$

Proof: For every elements $A_{(i,j)}, i, j \in \{1, \ldots, n\}$, of the real matrix A we have:

- whether $A_{(i,j)} \geq 0$, which implies that $A_{P(i,j)} \geq A_{N(i,j)}$. In addition, by definition we have

  $$A^+_{(i,j)} = A_{(i,j)} = A_{P(i,j)} - A_{N(i,j)}$$

  Since both $A_{P(i,j)}$ and $A_{N(i,j)}$ are nonnegative, one can state that $A_{P(i,j)} \geq A^+_{(i,j)}$.
- or $A_{(i,j)} \leq 0$, which implies that $A_{P(i,j)} \leq A_{N(i,j)}$. Moreover, by construction we have

  $$A^-_{(i,j)} = |A_{(i,j)}| = -(A_{P(i,j)} - A_{N(i,j)}) = A_{N(i,j)} - A_{P(i,j)}$$

  Also in this case, since both $A_{P(i,j)}$ and $A_{N(i,j)}$ are nonnegative, one can claim that $A_{N(i,j)} \geq A^-_{(i,j)}$.

This ends the proof. ■

C. Motivating Problem

Let A and C be two real matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{p\times n}$, respectively, such that the pair (A, C) is detectable. The studied problem in this subsection is formulated as follows:

Problem 1: Does it exist a real matrix $L \in \mathbb{R}^{n\times p}$ such that the positive system

$$\begin{pmatrix} \underline{e}_{k+1} \\ \bar{e}_{k+1} \end{pmatrix} = H \begin{pmatrix} \underline{e}_k \\ \bar{e}_k \end{pmatrix} \tag{4}$$

where $H = \begin{pmatrix} M^+ & M^- \\ M^- & M^+ \end{pmatrix}$ and $M = A - LC$, is asymptotically stable? To solve this problem, we propose the following sufficient condition.

Theorem 1: If there exist a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$, two nonnegative matrices $G_P, G_N \in \mathbb{R}^{n\times n}_{\geq 0}$, a real matrix $L \in \mathbb{R}^{p\times n}$ and a positive real scalar β such that

$$\begin{bmatrix} -P + \beta I_{2n} & G^\top P \\ \star & -P \end{bmatrix} \preceq 0 \tag{5}$$

$$G_P - G_N = A - LC \tag{6}$$

where

$$G = \begin{pmatrix} G_p & G_N \\ G_N & G_P \end{pmatrix},$$

Then, the augmented matrix H computed from $M = A - LC$ is Schur stable.

Proof: First, using the Schur complement, one can claim that the BMI condition (5) is equivalent to the discrete Lyapunov inequality

$$G^T P G - P \preceq -\beta I_{2n} \tag{7}$$

Thus, if (7) is true then the matrix G is schur stable. That is,

$$\rho(G) = \max_{\lambda \in Sp(G)} |\lambda| < 1 \tag{8}$$

where $Sp(G)$ stands for the set of the eigenvalues of G. Moreover, by construction, the matrices G and H are nonnegative and based on (6) and Proposition 2 one can affirm that:

$$H \leq G \tag{9}$$

Consequently, using the properties of positive matrices one obtains for all $k \geq 1$:

$$\begin{array}{rcl} \|H^k\|_\infty & \leq & \|G^k\|_\infty \\ \|H^k\|_\infty^{\frac{1}{k}} & \leq & \|G^k\|_\infty^{\frac{1}{k}} \\ \lim\limits_{k\longrightarrow+\infty} \|H^k\|_\infty^{\frac{1}{k}} & \leq & \lim\limits_{k\longrightarrow+\infty} \|G^k\|_\infty^{\frac{1}{k}} \\ \rho(H) & \leq & \rho(G) \end{array} \tag{10}$$

That is, the Schur stability of G implies that of H, which guarantees the asymptotic stability of the positive system (4). This completes the proof. ■

Based on these corner stone results, we propose in next sections design methods of interval observers for switched discrete-time linear systems without using similarity transformations.

III. INTERVAL OBSERVER FOR SWITCHED LINEAR SYSTEMS

Consider discrete-time switched systems described by

$$\begin{cases} x(k+1) = A_\sigma x(k) + B_\sigma u(k) + w(k), \\ y(k) = C_\sigma x(k) + v(k), \end{cases} \quad k \in \mathbb{N}, \sigma \in \mathscr{I} \tag{11}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are respectively its state, input and output vector. $\sigma : \mathbb{R}_{\geq 0} \to \mathscr{I}$ is an arbitrary right-continuous piece-wise constant switching signal. Note that, the index set $\mathscr{I} = \{1, \ldots, q\} \subset \mathbb{N}$ is a finite set whose elements indicate the subsystems of the switched system (11). $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are respectively the state disturbance and the measurement noise vector.

The objective of this work is to design an interval observer for this class of systems. That is to design two coupled dynamical systems providing component-wise upper and lower bounds on the actual state vector of system (11), while ensuring the practical stability of the distance between these two bounds. To achieve that, we propose a method based on the unknown but bounded errors assumption.

Assumption 1: Let $\bar{v}(k)$, $\overline{w}(k)$ and $\underline{w}(k)$ be in $\mathscr{L}_\infty^p$ and $\mathscr{L}_\infty^n$, respectively, such that: $\forall k \in \mathbb{N}$

$$|v(k)| \leq \bar{v}(k), \;\; \underline{w}(k) \leq w(k) \leq \overline{w}(k). \tag{12}$$

Assumption 2: There exist two known vectors $\underline{x}(k_0)$, $\bar{x}(k_0) \in \mathbb{R}^n$ such that the unknown initial vector $x(k_0)$ of (11) satisfies

$$\underline{x}(k_0) \;\leq x(k_0) \leq \bar{x}(k_0).$$

Notice that Assumptions 1 and 2 are common in set-valued estimation literature.

A. IO Structure

One of the objectives of this work is to design tractable interval observers that are less demanding in terms of online computational resources. For this, we propose the following structure as a framer for the class of systems described in (11),

$$\begin{cases} \underline{x}(k+1) = \underline{A}_\sigma^+ \underline{x}(k) - \underline{A}_\sigma^- \bar{x}(k) + B_\sigma u(k) + \underline{w}(k) \\ \qquad\qquad - |\underline{L}_\sigma| \bar{v}(k) - \underline{L}_\sigma y(k), \\ \bar{x}(k+1) = \overline{A}_\sigma^+ \bar{x}(k) - \overline{A}_\sigma^- \underline{x}(k) + B_\sigma u(k) + \overline{w}(k) \\ \qquad\qquad + |\overline{L}_\sigma| \bar{v}(k) - \overline{L}_\sigma y(k), \end{cases} \quad \sigma \in \mathscr{I} \tag{13}$$

where $\underline{A}_\sigma = (A_\sigma + \underline{L}_\sigma C_\sigma)$, $\overline{A}_\sigma = (A_\sigma + \overline{L}_\sigma C_\sigma)$ and $\underline{L}_\sigma$, $\overline{L}_\sigma \in \mathbb{R}^{n\times p}$ are the observer gains to be computed such that (13) must have the following properties:

- Property 1 (*framing*): for all $k \geq k_0$ one has,

$$\underline{x}(k) \leq x(k) \leq \bar{x}(k) \tag{14}$$

- Property 2 (*practical stability*): there exists a positive constant c such that,

$$\forall \left\|\bar{x}(k_0) - \underline{x}(k_0)\right\|_\infty \in \mathbb{R}_{\geq 0}, \;\; \lim_{k\to+\infty} \left\|\bar{x}(k) - \underline{x}(k)\right\|_\infty \leq c \tag{15}$$

To show that, we have to study the dynamics of the estimation errors. Using the output equation in (11), the state equation of this system can be re-written for all $k \in \mathbb{N}$ and $\sigma \in \mathscr{I}$ as follows:

$$\begin{aligned} x(k+1) =& A_\sigma x(k) + B_\sigma u(k) \\ & + w(k) + L_\sigma^\bullet [C_\sigma x(k) + v(k) - y(k)] \\ =& (A_\sigma + L_\sigma^\bullet C_\sigma) x(k) \\ & + B_\sigma u(k) + w(k) + L_\sigma^\bullet [v(k) - y(k)] \\ =& [(A_\sigma + L_\sigma^\bullet C_\sigma)^+ - (A_\sigma + L_\sigma^\bullet C_\sigma)^-] x(k) \\ & + B_\sigma u(k) + w(k) + L_\sigma^\bullet [v(k) - y(k)] \end{aligned} \tag{16}$$

where $L_\sigma^\bullet \in \{\underline{L}_\sigma, \; \overline{L}_\sigma\}$. Thus, the dynamics of both lower and upper estimation error $\underline{e}(k) = x(k) - \underline{x}(k)$ and $\bar{e}(k) = \bar{x}(k) - x(k)$, respectively, are governed by the following coupled dynamical system:

$$\begin{cases} \underline{e}(k+1) = \underline{A}_\sigma^+ \underline{e}(k) + \underline{A}_\sigma^- \bar{e}(k) + (w(k) - \underline{w}(k)) \\ \qquad\qquad + (\underline{L}_\sigma v(k) + |\underline{L}_\sigma| \bar{v}(k)) \\ \bar{e}(k+1) = \overline{A}_\sigma^+ \bar{e}(k) + \overline{A}_\sigma^- \underline{e}(k) + (\overline{w}(k) - w(k)) \\ \qquad\qquad + (|\overline{L}_\sigma| \bar{v}(k) - \overline{L}_\sigma v(k)) \end{cases} \sigma \in \mathscr{I} \tag{17}$$

Due to the coupling between the upper and lower estimation error, we define by $\xi = [\underline{e}^\top, \bar{e}^\top]^\top$ the augmented error vector of the interval observer (13). Thus, the dynamics in (17) can be written in a compact form

$$\xi(k+1) = \Lambda_\sigma \xi(k) + \Psi_\sigma(k), \quad \forall \sigma \in \mathscr{I} \tag{18}$$

where

$$\Lambda_\sigma = \begin{bmatrix} \underline{A}_\sigma^+ & \underline{A}_\sigma^- \\ \overline{A}_\sigma^- & \overline{A}_\sigma^+ \end{bmatrix} \tag{19}$$

and

$$\Psi_\sigma(k) = \begin{bmatrix} w(k) - \underline{w}(k) \\ \overline{w}(k) - w(k) \end{bmatrix} + \begin{bmatrix} \underline{L}_\sigma v(k) + |\underline{L}_\sigma| \bar{v}(k) \\ |\overline{L}_\sigma| \bar{v}(k) - \overline{L}_\sigma v(k) \end{bmatrix}. \tag{20}$$

B. Positivity and Stability Analysis

In this subsection, we introduce sufficient conditions that allow to prove that the proposed estimation structure (13) has the sought properties (14) and (15).

Theorem 2: Suppose that Assumptions 1 and 2 hold. For given gains matrices $\underline{L}_\sigma$, $\overline{L}_\sigma \in \mathbb{R}^{n\times p}$ and a positive constant δ, if there exist a positive semi-definite matrix $P \in \mathbb{R}^{2n\times 2n}$ and a positive scalar β such that

$$\begin{bmatrix} -P + \beta I_{2n} & \Lambda_\sigma^\top P \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0 \quad \forall \sigma \in \mathscr{I} \tag{21}$$

then the estimation structure (13) is an interval observer for the discrete-time switched linear system (11) with the properties (14) and (15).

Proof: The proof is divided into two parts. We start by presenting the proof of the framing property (14) and then we give the proof of the stability property (15).

Positivity of the estimation errors: Note that, inequalities (12) in Assumption 1 imply that, by construction, the vector $\Psi_\sigma(k)$ is nonnegative for all $k \in \mathbb{N}$ and $\sigma \in \mathscr{I}$. On the other hand, by definition the matrix Λ_σ are nonnegative for all $\sigma \in \mathscr{I}$. Thus, the dynamical system described in (18) is a positive system. That is, the augmented vector $\xi(k)$ of the estimation error is nonnegative for all $k \geq k_0$ provided that $\xi(k_0)$ is nonnegative (property (1) of positive systems). In other terms, both upper and lower estimation errors $\overline{e}(k)$ and $\underline{e}(k)$ are nonnegative. Consequently, the solutions to (13) preserve, for all $k \geq k_0$, the order relationship $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$. That is the framing property (14).

Practical stability: To analyze the stability of the switched system (18), we use a quadratic Lyapunov function,

$$V(\xi) = \xi^\top P \xi \tag{22}$$

The discrete-time variation of $V(\xi)$ is expressed as follows

$$\Delta V(\xi) = \xi^\top \Lambda_\sigma^\top P \Lambda_\sigma \xi + 2\xi^\top \Lambda_\sigma^\top P \Psi_\sigma - \xi^\top P \xi \tag{23}$$

For any positive scalar δ, by applying the Young's inequality one obtains,

$$\Delta V(\xi) \leq \xi^\top \left[(1+\delta)\Lambda_\sigma^\top P \Lambda_\sigma - P\right]\xi + (1+\frac{1}{\delta})\Psi_\sigma^\top P \Psi_\sigma$$

Now, by using Schur's complement, one can claim that (21) is equivalent to

$$(1+\delta)\Lambda_\sigma^\top P \Lambda_\sigma - P \preceq -\beta I_{2n},\ P \succeq 0 \tag{24}$$

Thus, one can upper bound the expression in (23) by,

$$\Delta V(\xi) \leq -\beta \xi^\top \xi + (1+\frac{1}{\delta})\Psi_\sigma^\top P \Psi_\sigma \tag{25}$$

By using similar arguments as those in [24, Definition 4.4], one can state that the augmented estimation error is Input-to-State Stable (ISS) relatively to the bounded vector Ψ_σ. Moreover, this estimation error is exponentially stable in the case where system (11) is free from state disturbances and measurement noises. Thus, we can claim that condition (21) guarantees the practical stability of the estimation error, that is property (15). ■

So far, we have shown if there exist observer gain matrices such that (21) is satisfied then the estimation structure (13) is an interval observer for system (11). To complete this study, in what follows, we propose two computing methods of these observer gains.

IV. OBSERVER GAINS DESIGN APPROACHES

In this section, we introduce two methods to compute the observer gains that ensure properties (14) and (15) for the estimation structure (13). More precisely, we show how to compute the observer gains in Theorem 2 without requiring any similarity transformation. The first method is based on the BMI sufficient condition (21), while the second one relies on an LMI condition.

1) BMI-based design method: The following theorem establishes a direct link between the BMI condition (21) and the observer gains allowing to the solutions of the dynamical structure (13) to have the desired properties (14) and (15).

Theorem 3: Let Assumptions 1 and 2 hold. For a given $\delta > 0$, if there exist symmetric positive definite matrix $P \in \mathbb{R}^{2n \times 2n}$, nonnegative matrices $G_{i,\sigma} \in \mathbb{R}^{n\times n}_{\geq 0}$, $i = \{1,\ldots,4\}$, two matrices $\underline{L}_\sigma$, $\overline{L}_\sigma$ and a positive scalar β such that: $\forall \sigma \in \mathscr{I}$

$$\begin{bmatrix} -P + \beta I_{2n} & G_\sigma^\top P \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0, \tag{26a}$$

$$G_{1,\sigma} - G_{2,\sigma} = A_\sigma + \underline{L}_\sigma C_\sigma, \tag{26b}$$

$$G_{4,\sigma} - G_{3,\sigma} = A_\sigma + \overline{L}_\sigma C_\sigma, \tag{26c}$$

with

$$G_\sigma = \begin{bmatrix} G_{1,\sigma} & G_{2,\sigma} \\ G_{3,\sigma} & G_{4,\sigma} \end{bmatrix}$$

then the estimation structure (13) is an interval observer for system (11) that owns the framing and converging properties (14) and (15).

Proof: First, based on the results of Theorem 2, the solution to the BMI condition (26a) allows one to design the following ISS stable coupled system

$$\begin{cases} \underline{z}(k+1) = G_{1,\sigma}\underline{z}(k) + G_{2,\sigma}\overline{z}(k) + (w(k) - \underline{w}(k)) \\ \qquad\qquad + (\underline{L}_\sigma v(k) + |\underline{L}_\sigma|\overline{v}(k)) \\ \overline{z}(k+1) = G_{4,\sigma}\overline{z}(k) + G_{3,\sigma}\underline{z}(k) + (\overline{w}(k) - w(k)) \\ \qquad\qquad + (|\overline{L}_\sigma|\overline{v}(k) - \overline{L}_\sigma v(k)) \end{cases} \sigma \in \mathscr{I} \tag{27}$$

On the other hand, using (26b)-(26b) and the results of Propositions 1 and 2, we can state that the time evolution of the estimation error (17) is always upper bounded by the solution to (27). That is,

$$\forall k \geq k_0,\ \ \underline{e}(k) \leq \underline{z}(k) \text{ and } \overline{e}(k) \leq \overline{z}(k) \tag{28}$$

Thus, based on Theorem 1, we can claim that the ISS stability of (27) implies that of (17). This ends the proof. ■

Remark 1: It is worth pointing out that the proposed design method in Theorem 3 is different from that introduced in [5] to deal with the case of discrete-time linear systems. In [5], the design is based on splitting the observer state matrix $M = A - LC$ into positive and negative parts (e.g., $M = M^+ - M^-$) which is a nonlinear operation due to the use of the max operator to compute the matrices M^+ and M^-. That is, M^+ and M^- are nonlinear expressions in the unknown gain L. However, in the proposed approach, we avoid the use of the *max* operator, in the synthesis step, by introducing intermediate nonnegative matrices (e.g., $G_{i,\sigma},\ i \in \{1,\ldots,4\}$.

Remark 2: After computing $\underline{L}_\sigma$ and $\overline{L}_\sigma$ by solving (26), the proposed interval observer should be implemented with its minimal realizations defined by the matrices $\overline{A}_\sigma^+$, $\overline{A}_\sigma^-$, $\underline{A}_\sigma^+$ and $\underline{A}_\sigma^-$ to get tighter state enclosures.

2) LMI-based design method: In this subsection, based on the properties of Stieltjes matrices, we propose an LMI condition to design interval observers for the discrete-time switched linear system (11).

Corollary 4: Assume that Assumptions 1 and 2 are true. For a given $\delta > 0$, if there exist two Stieltjes matrices $P_1, P_2 \in \mathscr{S}^{n\times n}$, nonnegative matrices $\Omega_{i,\sigma}$, $i = \{1,\ldots,4\}$, two matrices $\underline{U}_\sigma$, $\overline{U}_\sigma$ and a positive scalar β such that: $\forall \sigma \in \mathscr{I}$

$$\begin{bmatrix} -P + \beta I_{2n} & \Omega_\sigma^\top \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0, \tag{29a}$$

$$\Omega_{1,\sigma} - \Omega_{2,\sigma} = P_1 A_\sigma + \underline{U}_\sigma C_\sigma, \tag{29b}$$

$$\Omega_{4,\sigma} - \Omega_{3,\sigma} = P_2 A_\sigma + \overline{U}_\sigma C_\sigma, \tag{29c}$$

where $\Omega_\sigma = \begin{bmatrix} \Omega_{1,\sigma} & \Omega_{2,\sigma} \\ \Omega_{3,\sigma} & \Omega_{4,\sigma} \end{bmatrix}$, $P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}$, then the estimation structure (13) with the gains $\underline{L}_\sigma = P_1^{-1}\underline{U}_\sigma$ and $\overline{L}_\sigma = P_2^{-1}\overline{U}_\sigma$ is an interval observer for system (11) that has the desired properties (14) and (15).

Proof: To be able to transform the BMI design problem (26) to an LMI one, we consider a special form for the Lyapunov function (22), where

$$P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}$$

and (P_1, P_2) are Stieltjes matrices. Based on the property of the inverse of a Stieltjes matrix introduced in Definition 1, we know that P_1^{-1} and P_2^{-1} are nonnegative matrices.
By by setting $\Omega_\sigma = PG_\sigma$, one can claim that BMI (26a) is equivalent to LMI (29a). On the other hand, one has

$$G_\sigma = \begin{bmatrix} G_{1,\sigma} & G_{2,\sigma} \\ G_{3,\sigma} & G_{4,\sigma} \end{bmatrix} = \begin{bmatrix} P_1^{-1}\Omega_{1,\sigma} & P_1^{-1}\Omega_{2,\sigma} \\ P_2^{-1}\Omega_{3,\sigma} & P_2^{-1}\Omega_{4,\sigma} \end{bmatrix}$$

Thus, the nonnegativity of Ω_σ and the property of the Stieltjes matrix P imply the nonnegativity of the matrix G_σ.
Pre-multiplying both sides of equality (29b) and (29c) by P_1^{-1} and P_2^{-1}, respectively, we obtain:

$$\begin{aligned} P_1^{-1}\Omega_{1,\sigma} - P_1^{-1}\Omega_{2,\sigma} &= A_\sigma + \underline{L}_\sigma C_\sigma, \\ P_2^{-1}\Omega_{4,\sigma} - P_2^{-1}\Omega_{3,\sigma} &= A_\sigma + \overline{L}_\sigma C_\sigma, \end{aligned} \tag{30}$$

Consequently, one can state that equations (30) are equivalent to constraints (26b)-(26c). This ends the proof. ■

V. Illustrative Example

To illustrate the effectiveness of the proposed interval state estimation approach for switched discrete-time linear systems, in this section we compare its performance, through a numerical example, with that provided by another method selected from the literature. Consider the discrete-time linear switched system introduced in Dinh et al. [16], of the form (11) without output noise ($v(k) = 0\ \forall k \in \mathbb{N}$). This switched system has three subsystems defined by the following matrices:

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3 & -2 \\ 0 & 0.6 \end{bmatrix},\ B_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.5 & -1.1 \\ 0 & 0.16 \end{bmatrix},\ B_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},\ C_3 = \begin{bmatrix} 0.1 & 1 \end{bmatrix}$$

In this study, we compare our design method based on Corollary 4 with that introduced in Dinh et al. [16]. For the sake of brevity, in what follows, our state estimation method will be denoted by **LMI-IO** while that in [16] will be denoted by **Optimal-IO-19**. In the case of **LMI-IO** approach, the observer gains are computed by solving the LMI-based problem (29) using the YALMIP toolbox [25] based on SeDuMi 1.3 solver. The obtained numerical values are:

$$\underline{L}_1 = \overline{L}_1 = \begin{bmatrix} 0.0645 \\ -0.1975 \end{bmatrix},\ \underline{L}_2 = \overline{L}_2 = \begin{bmatrix} -0.2967 \\ -0.0012 \end{bmatrix},$$

$$\underline{L}_3 = \overline{L}_3 = \begin{bmatrix} 0.6974 \\ -0.1512 \end{bmatrix} \text{ with}$$

$$P_1 = \begin{bmatrix} 0.4112 & -0.0124 \\ -0.0124 & 3.6098 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.4112 & -0.0124 \\ -0.0124 & 3.6098 \end{bmatrix}.$$

On the other hand, the optimal observer gains used by **Optimal-IO-19** method are:

$$L_1 = \begin{bmatrix} -.026 \\ 0.0914 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.5416 \\ -0.0758 \end{bmatrix},\ L_3 = \begin{bmatrix} -0.857 \\ 0.1028 \end{bmatrix},$$

and the applied similarity transformation matrices T_σ that make $T_\sigma(A_\sigma - L_\sigma C_\sigma)T_\sigma^{-1}$ nonnegative $\forall \sigma \in \{1,2,3\}$ are given by

$$T_1 = \begin{bmatrix} 0.0901 & 0.6930 \\ -0.0901 & 0.3070 \end{bmatrix},\ T_2 = \begin{bmatrix} -0.2372 & -0.8173 \\ 0.2372 & 1.8173 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0.0191 & 0.9913 \\ -0.0191 & 0.0087 \end{bmatrix}.$$

The same simulation conditions as that used in [16] are considered in this study. That is,

- Initial conditions are set as follows,
$$x_0 = [1,2]^\top,\ \overline{x}_0 = [1.5,\ 2.5]^\top,\ \underline{x}_0 = [0.5, 1.5]^\top$$
- State disturbance and its lower and upper bounds are given by
$$\underline{w}(k) = -\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \le w(k) = \begin{bmatrix} 0.1\sin(0.5k) \\ 0.1\cos(0.5k) \end{bmatrix} \le \overline{w}(k) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$
- Switching signal is illustrated in Figure 1.
- System input is a sinusoidal signal $u = 0.5\sin(0.1k)$

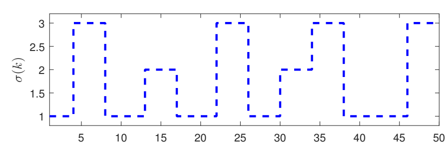

Fig. 1. Switching signal.

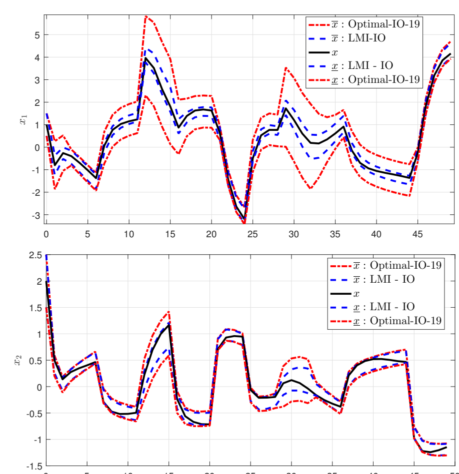

Fig. 2. Interval state estimates. Blue dashed lines correspond to **LMI-IO** method while the dash-dotted lines correspond to **Optimal-IO-19** method. Solid lines stand for actual state variables.

The simulation results are plotted in Figures 2 and 3. The upper and lower estimated bound of the state variables provided by both methods (**LMI-IO** and **Optimal-IO-19**) are plotted in Figure 2. Furthermore, in order to compare clearly the performance of these two approaches, we have also plotted

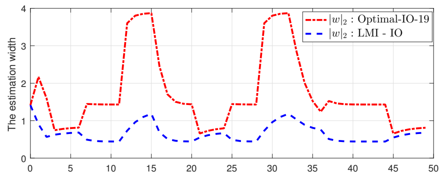

Fig. 3. Widths of the estimated interval: Blue dashed line (**LMI-IO** method). Red dash-dotted line (**Optimal-IO-19** method).

the 2−norm of the width of the interval estimate, defined as $|w|_2 = |\overline{x} - \underline{x}|_2$, of each approach in Figure 3. From the curves in this figure, one can deduce that **LMI-IO** method provides tighter interval estimate than **Optimal-IO-19** method.

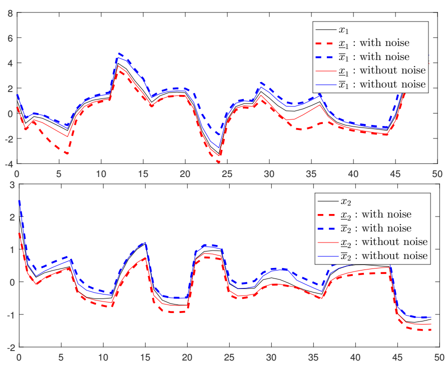

Fig. 4. Interval estimates of the approach **LMI-IO** of Corollary 4 in the presence of measurement noise (dashed) and without noise (solid).

So far, we have compared our approach to the one in [16], where the measurement noise was not considered. Thus, in the next experiment, we consider the case of noisy measurements. Figure 4 shows the interval state estimate obtained by **LMI-IO** method, where the system output is affected by a bounded noise. The considered unknown noise is $v(k) = 0.5\cos(0.2k)$ with known upper bound $\overline{v}(k) = 0.5$. The other simulation conditions are kept the same as those used in the first experiment. Figure 4 illustrates the effectiveness of the proposed estimation approach despite the presence of noise in the available measurements.

VI. CONCLUSIONS

In this contribution, sufficient conditions have been proposed to design interval observers for switched discrete-time linear systems. Two design approaches have been introduced: (*i*) BMI-based condition; (*ii*) LMI-based condition. Based on the internal positivity representation of dynamical system, the interval observers are directly designed in the original state coordinates and thus the use of conservative similarity transformations is avoided. In terms of online computational effort, the introduced structure for switched interval observers is less demanding compared to the existing methods in the literature. In addition, the proposed methods address naturally the case of corrupted outputs by additive unknown-but-bounded measurement noise. Through a numerical example, we have shown the effectiveness of the proposed approach. In future works, we investigate the possibility to use other forms of Lyapunov functions, dedicated to positive systems, to relax further the existence and design conditions of the interval observers.

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