Control strategies for meeting time and capacity constraints in manufacturing plants based on discrete event systems

S. Bouazza, and S. Amari

Abstract— **This paper addresses the control problem related to marking and strict timing constraints in Timed Event Graphs (TEGs). An analytic approach is developed to design state feedback control laws using Min-Plus formalism. In this approach, the dynamic behavior of TEGs and the mixed constraints are represented by linear equations and inequalities within the Min-Plus algebra dioid. Utilizing these mathematical frameworks, control laws are formulated to enforce the required constraints on TEG paths, provided certain sufficient conditions are met. The proposed controllers are implemented through a set of monitor places that oversee the initial TEGs, ensuring all mixed constraints are satisfied. Finally, a manufacturing system case study is presented to demonstrate the theoretical results and validate the effectiveness of the proposed methods.**

Index Terms— Discrete event systems, Timed event graphs, Min-Plus algebra, capacity and temporal constraints, Control laws, manufacturing system.

I. Introduction

A manufacturing system subject to temporal and capacity constraints requires careful management of time and resources. To define such systems, you need to consider both time-based factors (like deadlines and production times) and resource-based factors (like machine and labor availability). Balancing these constraints can be done through effective scheduling, resource management, control and optimization techniques. These systems are typically modeled as Discrete Event Systems (DES), where state changes occur at specific times triggered by discrete events. The discrete nature of these systems enables precise monitoring and control, making them highly adaptable to dynamic industrial environments. However, managing these constraints requires advanced planning, control and real-time decision-making to balance productivity and compliance. As a consequence, researchers and practitioners are focusing on the development of control strategies to improve system performance while respecting imposed constraints.

Numerous studies have explored different types of restrictions, including those derived from temporal specifications [1-6]. Other research has focused on capacity limitations defined through marking constraints [7-11]. The first constraint is to imposes a maximum duration to complete a task or achieve a goal, ensuring adherence to the specified deadlines. The second one limits the maximum amount that a system or process can hold or process at any given time, ensuring the available resources are respected.

Several studies have explored the issue of capacity constraints, in particular, the work presented in [6] that proposed an optimal-supervisor approach to prevent the emergence of generalized Mutual Exclusion Constraints (MECs) in a maximally permissive in temporal Petri nets model. This approach was based on a partially modified state class graph, which was constructed by exclusively enabling uncontrolled transitions. However, this method negatively impacted system performance, leading to potential deadlock states. Building on these results, the study in [5] introduced a strategy to enforce a set of generalized MECs while ensuring deadlock-freeness in Timed Petri Nets (TPNs) with uncontrolled transitions. More recently, the authors in [11] applied a similar class of DES to address MEC-related challenge. The theories of TPNs and linear programming are both utilized in the design of supervisory controllers, addressing both logical and temporal requirements. However, since this method relies on a worst-case analysis, there is a possibility that no feasible solution exists under the given conditions.

Other work has modelled these constraints, particularly in the context of DES, using Time Event Graphs (TEGs) [12] and dioid algebra, an approach that has been well established in research for many years [13]. Dioid algebra offers a robust mathematical framework for developing linear models applicable to timed Petri nets in this context [7]. This area of study remains an active field of investigation and it is utilized to address a wide range of control challenges. Like the works of [2,4] that introduced an analytical approach utilizing TEGs and inequalities within Min-Plus algebra to ensure compliance with marking restrictions. This approach ensures that the maximum capacity (maximum number of tokens) in a path in the TEG is not exceeded. Authors in [3] extended the previous works on considering restrictions subject on pathways in TEGs with uncontrolled transitions. Other works [1] have proposed a state feedback control strategy for designing networks of TEGs that are subject to mutual exclusion constraints, where the controlled places offered multiple pathways.

The main objective of controller computation is to ensure that the constraints imposed are satisfied. Other studies have focused on devising control strategies to fulfill other types of

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requirements, known as times constraints, especially by respecting upper time limits in critical operational contexts to maintain efficiency and reliability. In this context, the authors [14] proposed a novel control synthesis technique for networked conflicting TEGs, aiming to satisfy upper temporal bounds specified for certain critical paths using Switching Max-Plus models. Similarly, the authors in [8,15] introduced a model based on Max-Plus formalism to design a feedback controller, ensuring that the system operates without breaching the time restrictions imposed on its states. Other works [10] have focused on output feedback control for Max-Plus linear systems, addressing temporal constraints defined as the maximum allowable elapsed time imposed on TEGs.

However, the preceding TEG-based methods often focus on specific marking constraints such as a fixed maximum number of tokens and timing constraints separately. Most of the studies reviewed in the literature have motivated us to design a new approach to tackle the challenge of mixed constraints in TEGs using an analytical framework. This method seeks to overcome the limitations identified in the preceding research. In contrast, our work expands the use of Min-Plus algebra to manage TEGs under mixed constraints, addressing both types simultaneously. Furthermore, these analytical techniques are adaptable and applicable to various fields, including flexible manufacturing systems, supply chains, production systems, and rail or urban networks. The main contributions of this work can be summarized as follows:

i) Propose an innovative method for describing TEG behavior under mixed constraints using Min-Plus linear equations. This involves formulating a linear inequality in Min-Plus algebra that encapsulates both marking and timing constraints. Specifically, these mixed constraints limit the number of tokens while imposing a maximum time threshold.

ii) We introduce formal techniques to design control laws that ensure compliance with mixed constraints in TEGs. These methods extend existing approaches [4,8] by jointly addressing temporal and capacity constraints.

The first part of this paper introduces the foundational concepts, focusing on DESs tools like Min-Plus algebra and TEGs with disturbed transitions. Section 3 addresses the current mixed constraints and its formulation within the framework of Min-Plus algebra. The following section outlines the core contribution of this work which is the design of the control law. In section 6, a process of a product in a manufacturing system is presented to demonstrate and apply the proposed method. Finally, the paper concludes with a summary of findings and suggestions for future research directions.

II. FUNDAMENTAL DEFINITIONS AND FORMALISMS

A. Min-Plus algebra

The Min-Plus dioid, also known as Min-Plus algebra and represented as $\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$, is organized in the following manner. The first law $\oplus$ refers to the min function, i.e. $\forall\, a, b \in \overline{\mathbb{R}}_{min}$: $a \oplus b = \min(a, b)$, with the neutral element $\varepsilon = +\infty$. The second one, $\otimes$ corresponds to classical addition, i.e. $\forall\, a, b \in \overline{\mathbb{R}}_{min}$: $a \otimes b = a + b$, with the neutral element $e = 0$. In matrix Min-Plus algebra, the element resulting from the intersection of the i^{th} row and j^{th} column of a matrix A is defined by A_{ij}. For $n, m \in \mathbb{N}$, the set of $n \times m$ matrices over $\overline{\mathbb{R}}_{min}$ is denoted by $\overline{\mathbb{R}}_{min}^{n\times m}$, and operations $\oplus$ and $\otimes$ are defined where $\forall A, B \in \overline{\mathbb{R}}_{min}^{n\times m}$, $\forall C \in \overline{\mathbb{R}}_{min}^{m\times p}$ as follows:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, \qquad \forall\, i = 1, \dots, n\ \forall\, j = 1, \dots, m.$$

$$(A \otimes C)_{ij} = \oplus_{k=1}^{m} A_{ik} \otimes C_{kj}, \quad \forall\, i = 1, \dots, n\ \forall\, j = 1, \dots, p.$$

Lemma [14]. The implicit equation ($x = A \otimes x \oplus B$) formulated within a complete dioid D, is solvable with x = ($A^* \otimes$ B) where A^* denotes the Kleene star of matrix A. This solution satisfies the following condition:

$$\forall A \in D, \qquad A^* = \oplus_{i\geq 0} A^i$$

where ($A^i = A^{i-1} \otimes$ A) and A^0= *I*. Note that *I* represent the unit matrix, where entries on the diagonal are e, and elsewhere they are ε.

B. Linear Min-Plus models for TEG

Min-Plus algebra is typically used for modeling and analyzing DES that involve synchronization, delays, and parallelism. In our context, this class of systems is represented by TEG. For each transition, we associate a counter function to quantify the number of firing events occurring at time *t*. TEG are a specialized form of Petri net, where each place has precisely one incoming transition and one outgoing transition with delays associated with places, and/or transitions [12]. In this study, we focus on TEGs where delays are associated with the places. In this graph, transitions are classified as either input or internal transitions. Input transitions, also referred to as source transitions, are unaffected by the firing of other transitions in the system. This work focuses specifically on TEG with one controllable input transition. In contrast, internal transitions are neither inputs nor outputs. The dynamic behavior of these TEG is modeled using Min-Plus algebra and linear equations. To achieve this, a counter function $\theta_i(t) \in \overline{\mathbb{R}}_{min}$ is assigned to each transition t_i. This function monitors the cumulative occurrences of transition t_i up to time t. As a result, the overall dynamics of this graph can be captured through an equation formulated in Min-Plus algebra as follow:

$$\theta(t) = \oplus_{\tau\geq 0} (A_\tau \otimes \theta(t-\tau) \oplus B_\tau \otimes u(t-\tau) \quad (1)$$

where, matrix $A_\tau \in \overline{\mathbb{R}}_{min}^{n\times m}$ encodes the initial marking of the TEG, where *n* designates the number of internal transitions of the graph and $n, m \in \mathbb{N}$, the set of $n \times m$ matrices over $\overline{\mathbb{R}}_{min}$. Each element M_{ij} represents a specific place p_{ij} in the TEG, where j and i are the input and output of this place, respectively. Matrix $B_\tau \in \overline{\mathbb{R}}_{min}^{n\times m}$ focuses on the controllable inputs' transitions of the TEG. The counter function of the controllable source transition at instant $(t-\tau)$ are present by $u\,(t-\tau)$. It is recognized that, for a live TEG, the state vector can be expanded and reformulated in the following explicit form, provided that all delays in the TEG are equivalent to one delay:

$$x(t) = A \otimes x(t-1) \oplus B \otimes u(t) \quad (2)$$

where A= $A_0^* \otimes A_1$, and $B = A_0^* \otimes B_0$. The matrix A_0^* is the Kleene star of A_0. Finally, for any integer τ such that $\tau \geq 1$, we obtain:

$$x(t) = A_\tau \otimes x(t-\tau) \oplus [\oplus_{K=0}^{\tau-1} A^k \otimes B \otimes u(t-k)] \quad (3)$$

III. Mixed constraint: Capacity and temporal simultaneously

Manufacturing systems are frequently subject to various capacity constraints, such as limited machine availability, labor shortages, or restricted material supply, as well as temporal constraint. The combination of these specifications is referred to as a mixed constraint. In this study, we examine a TEG where all transitions are uncontrollable, except for the input transition t_u. In this graph, we have a path ρ subject to a mixed condition, with *b* representing the maximum number of tokens along this path. The transitions t_j and t_i mark the input and output of ρ, respectively, while x_j and x_i define its contours function. The $M_\rho(t)$ notation denotes the number of tokens on path ρ at time *t*, and τ_ρ^{max} represents the maximum delay for path ρ. The constraints related to the marking and timing are formalized by the following inequalities:

$$\begin{cases} x_j(t) \otimes (b - M_{\rho 0}) \leq x_i(t) & (5) \\ M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq x_i(t) & (6) \end{cases}$$

These inequalities can be written by:

$$\begin{cases} x_j(t - \tau_\rho^{max}) \leq (b - M_{\rho 0}) \otimes x_i(t - \tau_\rho^{max}) \\ M_{\rho 0} x_j(t - \tau_\rho^{max}) \leq x_i(t) \end{cases}$$

We reduce these inequalities to obtain the following expressions:

$$\begin{cases} M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq b \otimes x_i(t - \tau_\rho^{max}) \\ M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq x_i(t) \end{cases}$$

We calculate the lower bound between these two constraints and represent it as a single equivalent constraint, which is expressed in Min-plus algebra as follows:

$$M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq b \otimes x_i(t - \tau_\rho^{max}) \oplus x_i(t) \quad (7)$$

The inequality presented earlier (7) can be rewritten as the system of inequalities (5) and (6), representing the time and marking constraints. This is referred to as the mixed constraint in the current research. The goal, therefore, is to develop state feedback control strategies that can simultaneously fulfill both constraints within the context of a timed event graph.

IV. Control synthesis

This section presents an approach utilizing Min-plus algebra to determine the control law that ensures compliance with the mixed constraints imposed on TEGs. Initially, we examine a TEG subject to a single constraint. Then, we extend the approach to multiple restrictions.

A. TEGs subject to one constraint

We consider a TEG that is subject to single mixed constraint applied to a specific path ρ. In this setup, t_j and t_i represent the input and output transitions, respectively. The counters associated with these transitions are defined as $x_j(t)$ and $x_i(t)$. The expression for the mixed constraint is formulated as follows:

$$M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq b \otimes x_i(t - \tau_\rho^{max}) \oplus x_i(t) \quad (8)$$

We consider that a path exists between the control transition t_u and transition t_j. This path is denoted by α and τ_α represents the total time sum of all places along this path. Consequently, we can express the following inequality:

$$x_j(t) \leq (A^{\tau_\alpha} \otimes B)_j \otimes u(t - \tau_\alpha) \quad (9)$$

Given that $\tau = \phi$ with $\phi \geq 1$, where ϕ is an integer, equation (3) can be expressed as follows:

$$x(t) = [A^\phi \otimes x(t-\phi)] \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes B) \otimes u(t-k)] \quad (10)$$

Then, $x_i(t)$ can be written as:

$$x_i(t) = \left[\oplus_{r=1}^{N} A_{ir}^\phi \otimes x_r(t-\phi)\right] \oplus \left[\oplus_{k=0}^{\phi-1} (A^k \otimes B)_i \otimes u(t-k)\right] \quad (11)$$

with $\phi \geq 1$, $A \in \overline{R}_{min}^{n \times n}$, *n* denote the number of internal transitions of TEG, and A_{ir}^ϕ is a i^{th} raw of A^ϕmatrix.

Theorem 1. Consider a TEG as represented by equation (2), which is subject to a mixed constraint outlined in (8), applied to the path ρ. Taking $\phi = \tau_\alpha + 1$, the control law can be written as:

$$u(t) = [\oplus_{r=1}^{N} F_{r1} \otimes x_r(t-1)] \oplus [\oplus_{r=1}^{N} F_{r2} \otimes x_r(t - 1 + \tau_\rho^{max})] \oplus [\oplus_{k=0}^{\phi-1} F_k \otimes u(t-k)] \quad (12)$$

With:

$$F_{r1} = \max(0, (b - M_{\rho 0}) - (A^{\tau_\alpha} \otimes B)_j \otimes A_{ir}^\phi)$$

$$F_{r2} = \max(0, (A_{ir}^\phi - M_{\rho 0}) - (A^{\tau_\alpha} \otimes B)_j)$$

$$F_k = \max(0, (b - M_{\rho 0} - (A^{\tau_\alpha} \otimes B)_j \otimes (A^k \otimes B)_i)$$

The causal control law that guarantees the fulfillment of constraint (6) can be defined if the following condition holds true.

$$M_{\rho 0} \otimes (A^{\tau_\alpha} \otimes B)_j \otimes u(t - \tau_\alpha) \leq \oplus_{k=0}^{\phi-1} b \otimes (A^k \otimes B)_i \otimes u(t - k + \tau_\rho^{max}) \quad (13)$$

Then the checked condition is:

$$(A^{\tau_\alpha} \otimes B)_j \leq (A^k \otimes B)_i - M_{\rho 0}, \text{for } k = 0 \text{ to } \tau_\alpha \quad (14)$$

Proof 1. We examine a TEG containing a path denoted by ρ, which is subject to a mixed constraint expressed as follows:

$$M_{\rho 0} \otimes x_j(t - \tau_\rho^{max}) \leq b \otimes x_i(t - \tau_\rho^{max}) \oplus x_i(t) \quad (15)$$

Since the path α connects the input transition t_u to the output transition t_j of pathway ρ, the following inequality is derived.

$$x_j(t) \leq (A^{\tau_\alpha} \otimes B)_j \otimes u(t-\tau_\alpha) \quad (16)$$

The equation for $x_i(t)$ can be expressed as:

$$x_i(t) = [\oplus_{r=1}^{N} A_{ir}^{\phi} \otimes x_r(t-\phi)] \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes \mathrm{B})_i \otimes u(t-\tau_\alpha) \quad (17)$$

By substituting equations (16) and (17) into the mixed constraint inequality, we obtain:

$$M_{\rho 0}(A^{\tau_\alpha} \otimes B)_j \otimes u\left(t-\tau_\alpha - \tau_\rho^{max}\right) \leq b\,[[\oplus_{r=1}^{N} A_{\mathrm{ir}}^{\phi} \otimes x_r\,(t-\phi-\tau_\rho^{max})] \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes B)_i \otimes u(t-k-\tau_\rho^{max})]] \oplus [\oplus_{r=1}^{N} A_{ir}^{\phi} \otimes x_r(t-1)] \oplus [\,(A^k \otimes B)_i \otimes u(t-k)] \quad (18)$$

The inequality (18) verifies the satisfaction of the following two conditions:

$$u(t-\tau_\alpha) \leq [\oplus_{k=0}^{\phi-1}\ (A^k \otimes \mathrm{B})_i \otimes u(t-k+\tau_\rho^{max})] \quad (19)$$

and

$$M_{\rho 0}(A^{\tau_\alpha} \otimes \mathrm{B})_j \otimes u(t-\tau_\alpha) \leq b[[\oplus_{r=1}^{N} A_{ir}^{\phi} \otimes x_r(t-\phi)] \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes \mathrm{B})_i \otimes u(t-k)]] \oplus [\oplus_{r=1}^{N} A_{ir}^{\phi} \otimes x_r(t-\phi+\tau_\rho^{max})] \quad (20)$$

For $\phi = \tau_\alpha + 1$, the control law $u(t)$ is defined as a function of $x(t-1)$:

$$u(t) \leq (\mathrm{b} - M_{\rho 0}) - (A^{\tau_\alpha} \otimes \mathrm{B})_j\,[\oplus_{r=1}^{N} A_{ir}^{\Phi} \otimes x_r(t-1) \oplus [\oplus_{k=0}^{\phi-1} (A^k \otimes \mathrm{B})_i \otimes u(t-k)]] \oplus [\oplus_{r=1}^{N} (A_{ir}^{\Phi} - M_{\rho 0}) - (A^{\tau_\alpha} \otimes B)_j \otimes x_r(\mathrm{t}-1+\ \tau_\rho^{max})] \quad (21)$$

We have $u(t) \geq u(t-1) \ldots \geq u(t-n)$, then the condition can be expressed as follows:

$$(A^{\tau_\alpha} \otimes \mathrm{B})_j \leq (A^k \otimes \mathrm{B})_i - M_{\rho 0} \quad (22)$$

This inequality explicitly represents the control law outlined in Theorem 1 and guarantees compliance with the mixed constraint. □

B. TEGs subject to multi constraints

In this subsection, we extend our work by considering TEGs subject to *S* constraints applied to *S* specific paths, denoted ρ_s for $s = 1$ to *S*. The initial marking of each path ρ_s is represented by $M_{0\rho s}$, and $\tau_{\rho s}^{max}$ denotes its maximum allowable delay. We define x_{js} and x_{is}, as the counters corresponding to the input and output transitions, respectively, while b_s represents the number of tokens within path ρ_s. This mixed constraint can be expressed as follows:

$$M_{0\rho s} \otimes x_{js}\left(t-\tau_{\rho s}^{max}\right) \leq b_s \otimes x_{is}\,(t-\tau_{\rho s}^{max}) \oplus x_{is}(t) \quad (23)$$

Theorem 2. Considering a TEG subject to several constraints, represented as inequality (7) applied to the trajectories *S*, we define the control law with the following equation:

$$u(t) = \oplus_{s>1}\ u_s(\mathrm{t}) \quad (24)$$

with:

$$u_s(t) = [\oplus_{r=1}^{N} F_{r1s} \otimes x_r(t-1)] \oplus [\oplus_{r=1}^{N} F_{r2s} \otimes x_r(t-1+\tau^{max})]\ \oplus [\oplus_{k=0}^{\phi-1} F_{ks} \otimes u(t-k)]$$

for:

$$F_{r1s} = \max\left(0, (\,b_s - M_{0\rho s}) - (A^{\tau_{\alpha s}} \otimes \mathrm{B})_{js} \otimes A_{ir}^{\Phi}\right)$$

$$F_{r2s} = \max\left(0, (A_{ir}^{\Phi} - M_{0\rho s}) - (A^{\tau_{\alpha s}} \otimes \mathrm{B})_{js}\right)$$

$$F_k = \max\left(0, (b_s - M_{0\rho s}) - (A^{\tau_{\alpha s}} \otimes \mathrm{B})_{js} \otimes (A^k \otimes \mathrm{B})_{is}\right.$$

where $\Phi = \tau_{\alpha s} + 1$ and $s = 1, \ldots, S$

This control law is satisfied if the following condition is true:

$$(A^{\tau_{\alpha s}} \otimes \mathrm{B})_{js} \leq (A^k \otimes \mathrm{B})_{is} - M_{\rho 0 s} \quad (25)$$

Proof 2. In Theorem 1, we present a control law that ensures adherence to mixed constraints. For a set of *S* mixed constraints, the control law $u_s(t)$ guarantees compliance with each of these constraints, provided that certain conditions are met. □

V. APPLICATIONS: PROCESS OF A PRODUCT IN A MANUFACTURING SYSTEM

In this section, an example of a product life cycle model [18] is proposed, illustrating the flow of materials through various stages. The Fig. 1 illustrates the various processes in a product life cycle, integrating the key stages from initial production to final disposal from a circular economy perspective. The process begins with the extraction of raw materials, which are then transformed into finished products through development and production stages. These products are subsequently distributed to end-users. After use, products can be recovered for redistribution, extending their lifespan by allowing for reuse or alternative applications. If products are to be reintroduced into the cycle, they go through a disassembly phase to separate components, followed by cleaning, inspection, and material treatment. Recovered parts can be reassembled to create new products, thus reinforcing the reuse loop. Finally, materials that cannot be repurposed are disposed of. The multiple arrows and connections indicate closed-loop flows between different stages, highlighting the goal of waste reduction and resource optimization in a sustainability-focused approach. This system is modeled by a TEG, as illustrated in Fig. 2 and Table 1 presented their events. The latter includes a control transition, t_u , and imposes both the temporal and marking constraints on pathway ρ. Here, t_4 serves as the input transition, while t_7 acts as the output transition for this pathway ρ. As the disassembly operation (modelled by P_{65}) takes longer than the redistribution operation, the truck may have to wait for some time before unloading the parts. To solve this problem, it is preferable that truck do not load components during the disassembly process. Specifically, the maximum number of tokens that can be present is 1 and the maximum allowable time for this path is τ_ρ^{max}= 5. This means that the truck and disassembly machine are limited to taking a maximum of one order in no more than 5-time units. The latter includes a control transition, t_u , and imposes both the temporal and marking constraints on pathway ρ. Here, t_4 serves as the input

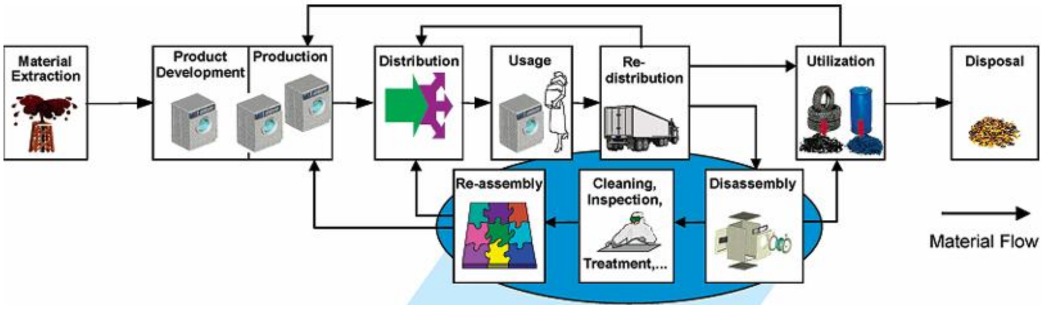

Figure 1. Process of a product in a manufacturing system [18].

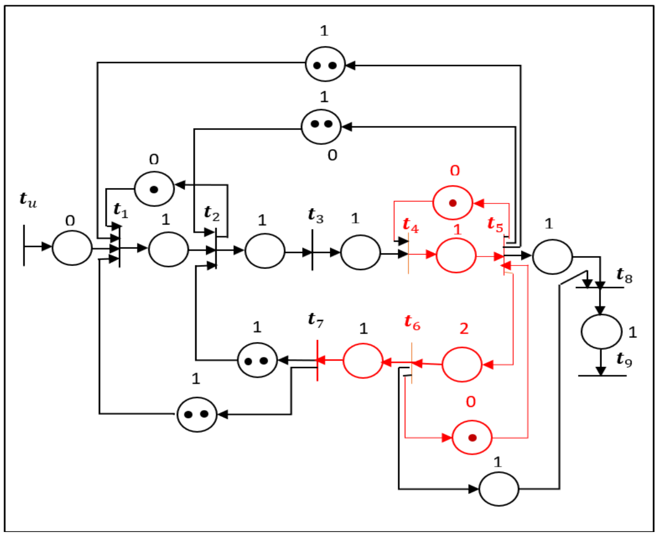

Figure 2. TEGs of the proposed system

transition, while t_7 acts as the output transition for this pathway ρ. As the disassembly operation (modelled by P_{65}) takes longer than the redistribution operation, the truck may have to wait for some time before unloading the parts. To solve this problem, it is preferable that truck do not load components during the disassembly process. Specifically, the maximum number of tokens that can be present is 1 and the maximum allowable time for this path is τ_ρ^{max}= 5. This means that the truck and disassembly machine are limited to taking a maximum of one order in no more than 5-time units. For this scenario, we assume that the input transition t_u is the only controllable transition. The place p_{65} timed is initially set to 2 times units, but it can be divided into two separate timed places, each representing 1 time unit.

The dynamic behavior of the system is then described by the following equation:

$$x(t) = A \otimes x(t-1) \oplus B \otimes u(t)$$

where:

$$A = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \text{ and } B = \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix},$$

Table 1. Modeling of the suggested example

Places	Description
p_{1u}	Preparing the material
p_{21}	Production process
p_{12}	Capacity of the production process
p_{32}	Distribution center
p_{23}	Capacity of the conveyor
p_{43}	Product usage
p_{54}	Redistribution
p_{45}	Capacity of the redistribution
p_{15}	Collection center for reuse
p_{25}	Collection center for repairs
p_{85}	Utilization
p_{98}	Disposal
p_{15}	Availability of buffer with capacity
p_{56}	Intermediate stock for components to be recycled
p_{65}	Disassembly process
p_{45}	Capacity of the disassembly process
p_{86}	Collection center for reuse
p_{76}	Cleaning and repair center
p_{27}	Collection center for send to redistribution
p_{17}	Collection center for send to production
Transition	**Description**
t_u	Arrival of end-of-life products
t_1	Send the material for workstations
t_2	Send to the distribution center
t_3	Transferring components to the first assembly station
t_4	Transferring components to the second assembly station
t_5	Sent for use or disassembly
t_6	Completion of parts disassembly and transfer to subsequent posts
t_7	Send material to distribution or production center
t_8	Transferring components to recycling
t_9	End of operation

The initial marking of the path ρ is equal to 0 ($M_{\rho 0} = 0$), then the inequality of the mixed constraint is:

$$x_4\,(t-5) \leq 1 \otimes\, x_7(t-5) \oplus x_7(t)$$

We assume that there is a path α that connects the input transition of TEG (t_u) and transition t_4, with timing $\tau_\alpha = 3$, then, $\phi = \tau_\alpha + 1 = 3 + 1 = 4$. We have also, $A_7^4 = [\varepsilon\ \varepsilon\ \varepsilon\ \ e\ \varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ \varepsilon]$, $(A^3 \otimes \mathrm{B})_4 = e$, $(A^k \otimes \mathrm{B})_7 = [\varepsilon, \varepsilon, \varepsilon, \varepsilon]$ Table 1. Identification of the elements used in the systems model shown in Figure 2 for $k = 0$ to 3. We can easily check that conditions of Theorem 2 is true. Then, by applying Theorem 2, we calculate the control law that satisfies the mixed constraint as follows:

$$u(t) = 1 - 0 - e \otimes e \otimes x_4(t-1) \oplus (1-e) \otimes x_4(t-1+5) \oplus (2-0) \otimes (A^k \otimes \mathrm{B})_7 - e \otimes u(\mathrm{t}-3)$$

$$= 1 \otimes x_4(t-1) \oplus 1 \otimes x_4(t+4) \oplus 1 \otimes u(t-3)$$

Taking into account the properties of the Min-plus algebra, and by comparing the terms, the control law reduces to:

$$u(t) = 1 \otimes x_4(t-1) \oplus 2 \otimes u(t-2)$$

This controller can be represented graphically by two control places, which are linked to the original TEG of the system to simultaneously enforce both the temporal and marking constraints. Figure 3 below shows the initial TEG with the integrated control places.

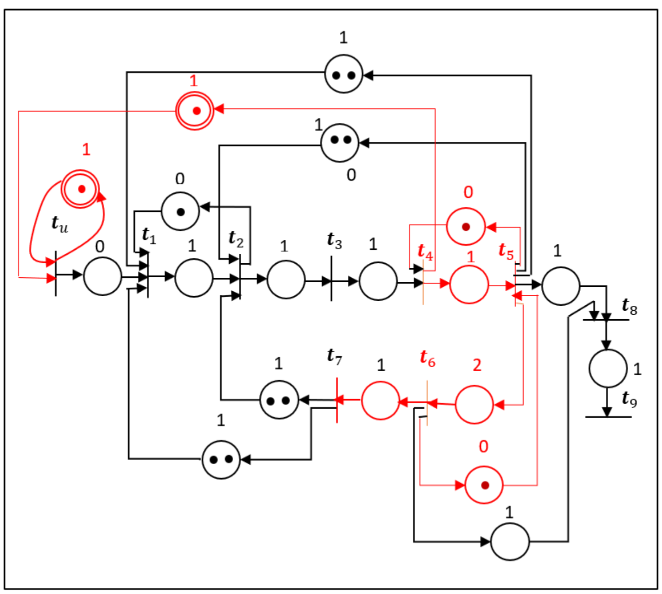

Figure 3. The controlled TEG of the manufacturing system

VI. Conclusion

This paper presents an analytic approach for designing control laws that ensure compliance with mixed constraints, including both temporal and marking restrictions, subject to a TEG. The method utilizes state feedback, defined by a set of marked and timed places, and is developed using Min-Plus algebra. The validity of these control laws is determined by verifying the sufficient condition based on the markings of the initial graph. The effectiveness of this control methodology is demonstrated through its application to a manufacturing system. Future research will focus on improving the robustness and optimality of the proposed control laws. Furthermore, the approach will be extended to broader classes of Petri nets, including colored and continuous Petri nets.

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