Optimal Control for Jump Diffusion Inventory Systems: Long-term Average Cost Criterion

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Abstract—This paper considers a long-term average inventory control problem, in which the inventory is subject to diffusion and compound Poisson demands. It establishes the optimality of an (s, S) ordering policy for the minimization of the long-term average total cost.

Keywords. Inventory control, (s, S) policy, long-term average control, quasi-variational inequality, verification theorem.

1. Introduction

This paper considers an inventory control problem where the underlying inventory is subject to diffusion and compound Poisson demands. The objective is to minimize the long-term average cost, which comprises three components: holding costs, shortage (or backordering) costs, and ordering costs. Holding costs represent the expenses associated with maintaining inventory. When inventory levels fall below zero, shortage or backordering costs are incurred due to unmet demand. Additionally, there is a positive control cost associated with restocking or adjusting inventory levels. The inclusion of positive ordering costs implies that the mathematical problem is of impulse control type. Hence, the optimal control policy involves discrete interventions, rather than continuous adjustments, to effectively manage inventory dynamics while minimizing overall costs.

Optimal inventory control problems have been extensively studied in the literature. Some early work on continuous time models can be found in Bather (1966), Harrison et al. (1983), Sulem (1986) and the references therein. More recently, Benkherouf and Bensoussan (2009), Bensoussan et al. (2005) established the optimality of an (s, S) policy under a discounted cost criterion for systems with compound Poisson and diffusion demand using quasi-variational inequalities approach. Similarly, Yamazaki (2017) proved the optimality of an (s, S) policy for a general spectrally positive Lévy demand process. The optimality of an (s, S) policy under a discounted criterion was also derived in Helmes et al. (2015) for state-dependent inventory process. In contrast, Christensen and Sohr (2020), He et al. (2017), Helmes et al. (2017, 2018, 2025), Yao et al. (2015) focused on optimal inventory control using long-term average cost criteria. We also refer to Bensoussan (2011) for a unified theory of dynamic programming and its applications in inventory control as well as Perera and Sethi (2023) for a comprehensive literature review on inventory control.

Notably, the majority of existing literature focuses on inventory control problems under discounted cost criteria. In contrast, studies analyzing long-term average costs primarily concentrate on continuous inventory processes. Research addressing long-term average inventory control for discontinuous inventory processes remains relatively scarce. An exception is Christensen and Sohr (2020), which develops a solution technique for a class of long-term average impulse control problems involving general Lévy processes. This paper seeks to investigate long-term average inventory control for a jump diffusion process using a different and more direct approach.

In our formulation, in the absence of control, the inventory system is governed by (2.1), where the Brownian motion accounts for the infinitesimal fluctuations in inventory levels, and the compound Poisson process captures sudden and discrete changes due to unexpected demand spikes or supply chain disruptions. The long-term average cost criterion (2.3) serves as a robust foundation for developing optimal policies over an extended time horizon, ensuring both stability and efficiency in managing the complex dynamics of such systems. To address problem (2.3), we extend the solution technique of Helmes et al. (2017), originally developed for long-term average inventory control in continuous inventory processes, to accommodate systems with jumps. Specifically, we begin by formulating a nonlinear optimization problem, which yields two critical thresholds, $y_* < z_*$. These thresholds not only facilitate the construction of a sufficiently smooth solution G to the quasi-variational inequality associated with (2.3), but also yield the optimal value for problem (2.3). Finally, we establish the optimality of the (y_*, z_*) policy through a verification theorem.

The paper is organized as follows. We begin with problem formulation in Section 2, and then solve the related quasi-variational inequality in Section 3. Section 4 establishes the optimality of the (y_*, z_*) policy. The paper concludes with several remarks in Section 5.

2. Problem Formulation

Our work builds upon the formulation in Bensoussan et al. (2005), which considers optimal inventory control

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with compound Poisson and diffusion demand under a discounted criterion. This paper focuses on the long-term average criterion. Suppose that in the absence of control, the inventory of a single item is modeled by

$$\begin{aligned} X_t^0 &= x - \mu t + \sigma W_t - N_t, \\ N_t &= \sum_{i=1}^{\mathfrak{P}_t} \xi_i \quad 0 \le t < \infty. \end{aligned} \tag{2.1}$$

where x is the initial inventory level, μ, σ are positive constants, W is a one-dimensional standard Brownian motion, $\mathfrak{P}$ is a Poisson process with rate $\lambda > 0$, and $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed exponential random variables with mean $\frac{1}{\rho} > 0$. We assume that W, $\mathfrak{P}$, and $\{\xi_n\}_{n=1}^{\infty}$ are independent. Put $\mathcal{F}_t := \sigma(W_s, N_s, 0 \le s \le t\}$ for each $t \ge 0$.

An admissible inventory control policy is a sequence $(\tau, Y) = \{(\tau_n, Y_n); n = 1, 2, \ldots\}$ in which $0 \le \tau_1 \le \tau_2 \le \ldots$ are $\{\mathcal{F}_{t+}\}$-stopping times and for each n, Y_n is nonnegative, $\mathcal{F}_{\tau_k}$-measurable. The dynamics of the inventory under an admissible control policy (τ, Y) is

$$X_t = x - \mu t + \sigma W_t - N_t + \sum_{k=1}^{\infty} Y_k I_{\{\tau_k < t\}}, \ 0 \le t < \infty. \tag{2.2}$$

The goal is to minimize the long-term average cost

$$\begin{aligned} J(\tau, Y) := \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\bigg[\int_0^t c_0(X_s)\mathrm{d}s \\ + \sum_{k=1}^{\infty}[c_1 + c_2 Y_k] I_{\{\tau_k < t\}}\bigg], \end{aligned} \tag{2.3}$$

where $c_0(x) := c_h x I_{\{x \ge 0\}} - c_b x I_{\{x<0\}}$, and c_h, c_b, c_1, and c_2 are positive constants. The quasi-variational inequality associated with (2.3) is

$$\begin{cases} Av(x) - c_0(x) + \eta \le 0, \\ v(x) \ge Mv(x) := \sup_{y > x}\{v(y) - c_1 - c_2(y - x)\}, \\ (Av(x) - c_0(x) + \eta)(v(x) - Mv(x)) = 0, \end{cases} \tag{2.4}$$

for all $x \in \mathbb{R}$, where v is a sufficiently smooth function and η is a constant, and the operator A is defined by

$$\begin{aligned} Af(x) := \frac{1}{2}\sigma^2 f''(x) - \mu f'(x) \\ + \lambda \int_0^{\infty} [f(x-y) - f(x)]\rho e^{-\rho y}\mathrm{d}y, \ f \in C^2(\mathbb{R}). \end{aligned}$$

Note that if (v, η) is a solution to (2.4), so is $(v + K, \eta)$ for any $K \in \mathbb{R}$.

3. Explicit Solution to the QVI

This section aims to find an explicit solution to (2.4). First, we note that the function $\psi(x) = \frac{\rho x}{\lambda + \rho\mu}$ solves the integro-differential equation

$$A\psi(x) = -1. \tag{3.1}$$

We next consider the integro-differential equation

$$Ag_0(x) = -c_0(x). \tag{3.2}$$

Denote $h(x) := \int_0^{\infty} g_0(x - y)\rho e^{-\rho y}\mathrm{d}y$. It is easy to verify that $h'(x) + \rho h(x) = \rho g_0(x)$. Therefore g_0 solves (3.2) if and only if it solves the following ordinary differential equation

$$\begin{aligned} \frac{1}{2}\sigma^2 g_0'''(x) + \left(\frac{1}{2}\sigma^2\rho - \mu\right) g_0''(x) \\ - (\rho\mu + \lambda)g_0'(x) = -c_0'(x) - \rho c_0(x), \end{aligned} \tag{3.3}$$

together with the condition

$$\begin{aligned} \frac{1}{2}\sigma^2 g_0''(0) - \mu g_0'(0) - \lambda g_0(0) \\ + \lambda\int_0^{\infty} g_0(-y)\rho e^{-\rho y}\mathrm{d}y = -c_0(0). \end{aligned} \tag{3.4}$$

The characteristic equation to (3.3)

$$\frac{1}{2}\sigma^2\beta^3 + \left(\frac{1}{2}\sigma^2\rho - \mu\right)\beta^2 - (\rho\mu + \lambda)\beta = 0$$

has three roots

$$\begin{aligned} \beta_1 := \frac{\mu}{\sigma^2} - \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2} + \frac{\mu}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}, \ 0, \\ \text{and } \beta_2 := \frac{\mu}{\sigma^2} - \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2} + \frac{\mu}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}. \end{aligned} \tag{3.5}$$

Note that $\beta_1 < -\rho < 0$ and $\beta_2 > 0$. Using these roots and taking into account of (3.4) and the smooth pasting at 0, detailed calculations reveal that

$$g_0(x) = \begin{cases} \frac{\sigma^2(c_b+c_h)(\rho+\beta_2)\beta_1^2}{2\beta_2(\lambda+\rho\mu)^2(\beta_2-\beta_1)}e^{\beta_2 x} \\ \quad -\frac{\rho c_b}{2(\lambda+\rho\mu)}x^2 - \frac{c_b(\rho^2\sigma^2+2\lambda)}{2(\lambda+\rho\mu)^2}x & \text{if } x < 0, \\ \frac{\sigma^2(c_b+c_h)(\rho+\beta_1)\beta_2^2}{2\beta_1(\lambda+\rho\mu)^2(\beta_2-\beta_1)}e^{\beta_1 x} + b_3 \\ \quad +\frac{\rho c_h}{2(\lambda+\rho\mu)}x^2 + \frac{c_h(\rho^2\sigma^2+2\lambda)}{2(\lambda+\rho\mu)^2}x & \text{if } x \ge 0. \end{cases} \tag{3.6}$$

is a solution to (3.2), where

$$b_3 = \frac{\sigma^2(c_b + c_h)(\rho + \beta_2)\beta_1^2}{2\beta_2(\lambda + \rho\mu)^2(\beta_2 - \beta_1)} - \frac{\sigma^2(c_b + c_h)(\rho + \beta_1)\beta_2^2}{2\beta_1(\lambda + \rho\mu)^2(\beta_2 - \beta_1)}.$$

Lemma 3.1. The function g_0 is twice continuously differentiable on $\mathbb{R}$. Moreover, if

$$1 - \frac{\beta_2(\rho^2\sigma^2 + 2\lambda)}{2\rho(\lambda + \rho\mu)} - \log\left(\frac{c_b\rho(\beta_2 - \beta_1)}{(c_b + c_h)(\rho + \beta_2)(-\beta_1)}\right) \ge 0. \tag{3.7}$$

then the function g_0 is increasing.

Proof. It is easy to see that $g_0 \in C(\mathbb{R})$. In addition, we have $g_0'(x) = k(x)I_{\{x\ge0\}} + h(x)I_{\{x<0\}}$, where

$$\begin{aligned} k(x) = \frac{\sigma^2(c_b + c_h)(\rho + \beta_1)\beta_2^2}{2(\lambda + \rho\mu)^2(\beta_2 - \beta_1)}e^{\beta_1 x} \\ + \frac{\rho c_h}{\lambda + \rho\mu}x + \frac{c_h(\rho^2\sigma^2 + 2\lambda)}{2(\lambda + \rho\mu)^2}, \quad x \ge 0 \end{aligned} \tag{3.8}$$

and

$$h(x) = \frac{1}{\lambda + \rho\mu}\left[\frac{\sigma^2(c_b + c_h)(\rho + \beta_2)\beta_1^2}{2(\rho\mu + \lambda)(\beta_2 - \beta_1)}e^{\beta_2 x} - \rho c_b x\right]$$

$$-\frac{c_b(\rho^2\sigma^2+2\lambda)}{2(\rho\mu+\lambda)^2}, \quad x<0. \tag{3.9}$$

Detailed computations reveal that $k(0) = h(0-)$ and $k'(0+) = h'(0-)$. This establishes the assertion that $g_0 \in C^2(\mathbb{R})$.

We now show that g_0 is increasing under condition (3.7). Since $\beta_1 < -\rho < 0 < \beta_2$, it follows that

$$k'(x) = \frac{\sigma^2(c_b+c_h)(\rho+\beta_1)\beta_1\beta_2^2}{2(\lambda+\rho\mu)^2(\beta_2-\beta_1)} e^{\beta_1 x} + \frac{\rho c_h}{\lambda+\rho\mu} > 0,$$

for $x \ge 0$. That is, k is strictly increasing. On the other hand, it is easy to see that h is strictly convex. Moreover, setting $h'(\tilde{x}) = 0$ yields that

$$\begin{aligned}\tilde{x} &= \frac{1}{\beta_2}\log\left(\frac{c_b}{c_b+c_h}\cdot\frac{2\rho(\lambda+\rho\mu)(\beta_2-\beta_1)}{\sigma^2(\rho+\beta_2)\beta_1^2\beta_2}\right)\\ &= \frac{1}{\beta_2}\log\left(\frac{c_b}{c_b+c_h}\cdot\frac{\rho(\beta_2-\beta_1)}{(\rho+\beta_2)(-\beta_1)}\right),\end{aligned} \tag{3.10}$$

where the last equality follows from the fact that $\sigma^2\beta_1\beta_2 = -2(\lambda+\rho\mu)$. Furthermore, since $\rho+\beta_1<0$ and $\beta_2>0$, we can readily verify that $0 < \frac{\rho(\beta_2-\beta_1)}{(\rho+\beta_2)(-\beta_1)} < 1$ and hence

$$\tilde{x} < \frac{1}{\beta_2}\log\frac{c_b}{c_b+c_h} < 0.$$

Therefore h is strictly decreasing in $(-\infty,\tilde{x}]$ and strictly increasing in $(\tilde{x},0)$. In other words, h and hence g_0' achieves their minimum value at $\tilde{x}$. Hence, in view of (3.7), we have for any $x\in\mathbb{R}$,

$$\begin{aligned}g_0'(x) \ge g_0'(\tilde{x}) = \frac{\rho c_b}{\beta_2(\lambda+\rho\mu)}\Bigg[1 - \frac{\beta_2(\rho^2\sigma^2+2\lambda)}{2\rho(\lambda+\rho\mu)}\\ - \log\left(\frac{c_b}{c_b+c_h}\cdot\frac{\rho(\beta_2-\beta_1)}{(\rho+\beta_2)(-\beta_1)}\right)\Bigg] \ge 0.\end{aligned}$$

This shows that g_0 is increasing. □

Remark 3.2. Note that when $\lambda=0$, the function g_0 is always increasing. In fact, when $\lambda=0$, the uncontrolled inventory (2.1) is a drifted Brownian motion

$$X_t^0 = x - \mu t + \sigma W_t, \quad t\ge 0,$$

and the function g_0 can be computed to be

$$g_0(x) = \begin{cases} -\frac{c_b}{2\mu}x^2 - \frac{c_b\sigma^2}{2\mu^2}x \\ \quad + \frac{\sigma^4(c_b+c_h)}{4\mu^3}\left(e^{2\mu x/\sigma^2}-1\right), & x<0,\\ \frac{c_h}{2\mu}x^2 + \frac{c_h\sigma^2}{2\mu^2}x, & x\ge 0.\end{cases}$$

Again, we can verify that $g_0\in C^2(\mathbb{R})$ and that g_0' achieves its minimum value at $\tilde{x} := \frac{\sigma^2}{2\mu}\log\frac{c_b}{c_b+c_h} < 0$ with

$$g_0'(\tilde{x}) = -\frac{c_b\sigma^2}{2\mu^2}\log\frac{c_b}{c_b+c_h} > 0.$$

Hence it follows that $g_0'(x) > 0$ for all $x\in\mathbb{R}$.

Proposition 3.3. Consider the function

$$F(y,z) = \frac{c_1 + c_2(z-y) + Bg_0(y,z)}{B\psi(y,z)}, \tag{3.11}$$

for $-\infty<y<z<\infty$. Then there exists a pair $y_* < z_*$ such that

$$F_* := F(y_*,z_*) = \inf_{y<z} F(y,z) > 0. \tag{3.12}$$

Moreover, the first order optimality condition holds true

$$F_* = F(y_*,z_*) = \frac{c_2+g_0'(y_*)}{\psi'(y_*)} = \frac{c_2+g_0'(z_*)}{\psi'(z_*)}. \tag{3.13}$$

Proof. Note that $F(y,z)\to\infty$ when $z-y\to 0$. For any y fixed, $F(y,z)\to\infty$ as $z\to\infty$ and similarly for any z fixed, $F(y,z)\to\infty$ as $y\to-\infty$. On the other hand, using the definition of the function ψ, we have

$$F(y,z) > \frac{\lambda+\rho\mu}{\rho}\cdot\frac{g_0(z)-g_0(y)}{z-y}.$$

Note also that $g_0'(x)\to\infty$ as $x\to\infty$ or $x\to-\infty$. Therefore it follows that $F(y,z)\to\infty$ when $(y,z)\to(\infty,\infty)$ or $(y,z)\to(-\infty,-\infty)$. Also it is clear from the expression for g_0 that $\frac{g_0(z)-g_0(y)}{z-y}\to\infty$ as $(y,z)\to(-\infty,\infty)$. Hence there exists a compact subset K of $\mathcal{R} := \left\{(y,z)\in\mathbb{R}^2 : y<z\right\}$ so that F achieves its minimum value at some $(y_*,z_*)\in K$. In addition, the first order optimality condition leads to (3.13), which, in turn, implies that $F_* \ge c_2\frac{\lambda+\rho\mu}{\rho} > 0$. □

Now let's consider the function

$$G(x) = \begin{cases} F_*\psi(x) - g_0(x), & \text{if } x\ge y_*,\\ G(z_*) - c_1 - c_2(z_*-x), & \text{if } x<y_*.\end{cases} \tag{3.14}$$

Using the definition of $F_* = F(y_*,z_*)$ and (3.13), we can readily verify that $G\in C^2(\mathbb{R}\setminus\{y_*\})\cap C^1(\mathbb{R})$. In fact, we have

$$\begin{aligned}G(y_*^-) &= G(z_*) - (c_1 + c_2(z_*-y_*))\\ &= F_*\psi(z_*) - g_0(z_*)\\ &\quad - (F_*[\psi(z_*)-\psi(y_*)] - [g_0(z_*)-g_0(y_*)])\\ &= F_*\psi(y_*) - g_0(y_*) = G(y_*^+);\end{aligned}$$

and from (3.13),

$$G'(y_*^+) = F_*\psi'(y_*) - g_0'(y_*) = c_2 = G'(y_*^-).$$

Proposition 3.4. The function G defined in (3.14) and the constant F_* of (3.12) together satisfy

$$\begin{cases} AG(x) - c_0(x) + F_* \le 0, & x\in\mathbb{R}\setminus\{y_*\},\\ G(z)-G(y)\le c_1+c_2(z-y), & y<z,\\ AG(x)-c_0(x)+F_* = 0, & x\in[y_*,\infty)\\ G(z_*)-G(x) = c_1 + c_2(z_*-x), & x\in(-\infty,y_*].\end{cases} \tag{3.15}$$

Therefore the pair (G,F_*) is a solution to the quasi-variational inequality (2.4).

Proof. First, we can use the definition of F_* and detailed computations to establish

$$\begin{aligned}&G(z)-G(y)\le c_1+c_2(z-y), \text{ for all } y<z,\\ &G(z_*)-G(x) = c_1+c_2(z_*-x), \text{ for all } x\le y_*.\end{aligned}$$

Therefore the second and fourth equations in (3.15) are established.

Next we show that

$$AG(x) - c_0(x) + F_* \le 0, \quad \text{for all } x \in \mathbb{R}. \tag{3.16}$$

Obviously (3.16) holds with equality for $x > y_*$ by the definitions of ψ, g_0, and G. It remains to show (3.16) for $x < y_*$. First we notice that since ψ is linear, the first order optimality condition (3.13) implies that $g_0'(y_*) = g_0'(z_*)$ for $y_* < z_*$. Recall from the proof of Lemma 3.1 that $g_0'(x) = k(x)$ for $x \ge 0$ and that k is strictly increasing. Hence the fact that $g_0'(y_*) = g_0'(z_*)$ for $y_* < z_*$ necessarily implies that y_* is negative.

Now using the the definition of G and fact that $y_* < 0$, it follows that for $x < y_*$, we have $AG(x) = -\frac{c_2(\lambda+\rho\mu)}{\rho}$. Using (3.13) and the definition of ψ, we have $F_* = \frac{c_2(\lambda+\rho\mu)}{\rho} + \frac{\lambda+\rho\mu}{\rho} g_0'(y_*)$. Hence for $x < y_* < 0$, we have

$$\begin{aligned} &AG(x) - c_0(x) + F_* \\ &= c_b(x - y_*) + \frac{1}{2\rho(\lambda+\rho\mu)} \\ &\times \left[\frac{\sigma^2(c_b+c_h)(\rho+\beta_2)\beta_1^2}{\beta_2-\beta_1} e^{\beta_2 y_*} - c_b(\rho^2\sigma^2 + 2\lambda)\right] \\ &\le \frac{1}{2\rho(\lambda+\rho\mu)} \left[\frac{\sigma^2(c_b+c_h)(\rho+\beta_2)\beta_1^2}{\beta_2-\beta_1} e^{\beta_2 y_*} \right. \\ &\qquad \left. - c_b(\rho^2\sigma^2 + 2\lambda)\right]. \end{aligned} \tag{3.17}$$

Recall that we have shown in Lemma 3.1 that g_0' achieves its unique minimum value at $\tilde{x} < 0$. Thus it follows from the facts that $g_0'(y_*) = g_0'(z_*)$ and $y_* < 0$ that $y_* < \tilde{x}$. But then we have

$$\begin{aligned} &\frac{\sigma^2(c_b+c_h)(\rho+\beta_2)\beta_1^2}{\beta_2-\beta_1} e^{\beta_2 y_*} - c_b(\rho^2\sigma^2 + 2\lambda) \\ &< \frac{\sigma^2(c_b+c_h)(\rho+\beta_2)\beta_1^2}{\beta_2-\beta_1} e^{\beta_2 \tilde{x}} - c_b(\rho^2\sigma^2 + 2\lambda) \\ &= c_b \left[\frac{2\rho(\lambda+\rho\mu)}{\beta_2} - (\rho^2\sigma^2 + 2\lambda)\right]. \end{aligned} \tag{3.18}$$

Recall that $\beta_2 > 0$ solves the equation

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2\beta_2^2 - \mu\beta_2 + \frac{\lambda\rho}{\beta_2+\rho} - \lambda \\ &= \frac{1}{2}\sigma^2\beta_2^2 - \mu\beta_2 + \lambda\int_0^\infty [e^{-\beta_2 y} - 1]\rho e^{-\rho y} \mathrm{d}y. \end{aligned}$$

Using the elementary inequality $e^{-x} - 1 \le \frac{x^2}{2} - x$, $x > 0$, we compute

$$\begin{aligned} 0 &\le \frac{1}{2}\sigma^2\beta_2^2 - \mu\beta_2 + \lambda\int_0^\infty \left(\frac{\beta_2^2 y^2}{2} - \beta_2 y\right)\rho e^{-\rho y}\mathrm{d}y \\ &= \frac{1}{2}\sigma^2\beta_2^2 - \mu\beta_2 + \lambda\left[-\frac{\beta_2}{\rho} + \frac{1}{2}\beta_2^2\frac{2}{\rho^2}\right] \\ &= \left[\frac{\sigma^2}{2} + \frac{\lambda}{\rho^2}\right]\beta_2^2 - \left[\mu + \frac{\lambda}{\rho}\right]\beta_2. \end{aligned}$$

Then it follows that $\beta_2 \ge \frac{\mu+\frac{\lambda}{\rho}}{\frac{\sigma^2}{2}+\frac{\lambda}{\rho^2}} = \frac{2\rho(\lambda+\rho\mu)}{\rho^2\sigma^2+2\lambda}$. This, together with (3.17) and (3.18), implies that $AG(x) - c_0(x) + F_* < 0$ or (3.16), as desired. □

4. Optimal Impulse Policy

We have found a solution (G, F_*) to (3.15) and hence (2.4). This section will prove that the (s, S) policy with $s = y_*$ and $S = z_*$ whose existence follows from Proposition 3.3 is optimal and that F_* is the optimal long-term average cost for problem (2.3).

Lemma 4.1. Let $(\tau, Y) \in \mathcal{A}$ and for $k = 1, 2, \ldots$, let τ_k denote the time of the kth order. Assume that

$$\begin{aligned} \mathbb{E}\left[f(X_t)\right] = f(x) + \mathbb{E}\bigg[&\int_0^t Af(X_s)\mathrm{d}s \\ &+ \sum_{k=1}^\infty [f(X_{\tau_k^+}) - f(X_{\tau_k})] I_{\{\tau_k < t\}}\bigg]. \end{aligned}$$

Then with $\tau_0 := 0$,

$$\mathbb{E}[f(X_t)] = \mathbb{E}\left[\sum_{j=0}^\infty I_{\{\tau_j < t \le \tau_{j+1}\}}\left[\int_{\tau_j}^t Af(X_s)ds + f(X_{\tau_j^+})\right]\right]. \tag{4.1}$$

Proof. Using the optional sampling theorem to justify the second equality below, we obtain for any $j \ge 0$,

$$\begin{aligned} &\mathbb{E}\left[I_{\{\tau_j < t \le \tau_{j+1}\}}[f(X_t) - f(X_{\tau_j^+})]\middle|\mathcal{F}_t\right] \\ &= I_{\{\tau_j < t \le \tau_{j+1}\}}\mathbb{E}\left[f(X_{t\wedge\tau_{j+1}}) - f(X_{t\wedge\tau_j^+})\middle|\mathcal{F}_t\right] \\ &= I_{\{\tau_j < t \le \tau_{j+1}\}}\mathbb{E}\left[\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} Af(X_s)\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[I_{\{\tau_j < t \le \tau_{j+1}\}}\int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} Af(X_s)\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[I_{\{\tau_j < t \le \tau_{j+1}\}}\int_{\tau_j}^{t} Af(X_s)\mathrm{d}s\middle|\mathcal{F}_t\right]. \end{aligned}$$

This together with

$$\begin{aligned} \mathbb{E}[f(X_t)] &= \mathbb{E}\left[\sum_{j=0}^\infty I_{\{\tau_j < t \le \tau_{j+1}\}} f(X_t)\right] \\ &= \mathbb{E}\left[\sum_{j=0}^\infty I_{\{\tau_j < t \le \tau_{j+1}\}}\left[f(X_t) - f(X_{\tau_j^+}) + f(X_{\tau_j^+})\right]\right] \end{aligned}$$

gives (4.1), as desired. □

Theorem 4.2. Assume condition (3.7). Let (τ, Y) be any admissible impulse control policy satisfying

$$\liminf_{t\to\infty}\liminf_{n\to\infty}\frac{1}{t}\mathbb{E}_x[G(X_{t\wedge\sigma_n})] \ge 0, \ \forall x \in \mathbb{R}. \tag{4.2}$$

where $\sigma_n = \inf\{t \ge 0 : |X_t| \ge n\}$, $n \in \mathbb{N}$, and X is the controlled process under (τ, Y). Then $F_* \le J(\tau, Y)$, where F_* is defined in (3.12). Moreover, the impulse policy defined by

$$\tau_1^* := \inf\{t \ge 0 : X_t^* \le y_*\}, Y_1^* := z_* - X_{\tau_1^*}^*, \tag{4.3}$$

and for $k \geq 1$,

$$\tau^*_{k+1} := \inf\{t \geq \tau^*_k : X^*_t \leq y_*\}, Y^*_{k+1} := z_* - X^*_{\tau^*_{k+1}} \tag{4.4}$$

is an optimal impulse policy, where y_* and z_* are as in (3.12).

Proof. The proof is divided into two steps.

Step 1. Let (τ, Y) be an arbitrary admissible impulse control policy in $\mathcal{A}$ so that the resulting controlled process X satisfies condition (4.2). We shall prove that F_* is a lower bound on $J(\tau, Y)$. Obviously it is enough to consider $(\tau, Y) \in \mathcal{A}$ with $J(\tau, Y) < \infty$.

We use a localization argument. For each $n \in \mathbb{N}$, define the stopping time σ_n as in the statement of the theorem. Then applying Dynkin's formula to the function G of (3.14) yields

$$\begin{aligned}
&\mathbb{E}_x[G(X_{t\wedge\sigma_n})] - G(x)\\
&= \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} AG(X_s)\mathrm{d}s + \sum_{k=1}^{\infty} I_{\{\tau_k < t\wedge\sigma_n\}} BG(X_{\tau_k}, X_{\tau_k^+})\right]\\
&\leq \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} [c_0(X_s) - F_*]\mathrm{d}s\right.\\
&\qquad \left. + \sum_{k=1}^{\infty} I_{\{\tau_k < t\wedge\sigma_n\}} (c_1 + c_2|X_{\tau_k^+} - X_{\tau_k}|)\right],
\end{aligned}$$

where the inequality follows from Proposition 3.4. Rearranging the terms and dividing both sides by t, it follows that

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}_x[F_*(t\wedge\sigma_n) + G(X_{t\wedge\sigma_n})]\\
&\quad \leq \frac{1}{t}G(x) + \frac{1}{t}\mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} c_0(X_s)\mathrm{d}s\right.\\
&\qquad \left. + \sum_{k=1}^{\infty} I_{\{\tau_k < t\wedge\sigma_n\}} (c_1 + c_2|X_{\tau_k^+} - X_{\tau_k}|)\right].
\end{aligned} \tag{4.5}$$

Clearly the sequence σ_n is nondecreasing. Denote $\sigma := \lim_{n\to\infty} \sigma_n$. If $\mathbb{P}\{\sigma < \infty\} > 0$, then $J(\tau, Y) = \infty$, resulting a contradiction to the assumption that $J(\tau, Y) < \infty$. Let's assume from now on that $\mathbb{P}\{\sigma = \infty\} = 1$. Therefore, letting $n \to \infty$ in (4.5), it follows from the Monotone Convergence Theorem that

$$\begin{aligned}
&F_* + \liminf_{n\to\infty} \frac{1}{t}\mathbb{E}_x[G(X_{t\wedge\sigma_n})]\\
&\quad \leq \frac{G(x)}{t} + \frac{1}{t}\mathbb{E}_x\left[\int_0^{t} c_0(X_s)\mathrm{d}s\right.\\
&\qquad \left. + \sum_{k=1}^{\infty} I_{\{\tau_k < t\}} (c_1 + c_2|X_{\tau_k^+} - X_{\tau_k}|)\right].
\end{aligned} \tag{4.6}$$

Passing to the limit as $t \to \infty$ in (4.6) and using (4.2), we obtain

$$\begin{aligned}
F_* &\leq \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}_x\left[\int_0^{t} c_0(X_s)\mathrm{d}s\right.\\
&\qquad \left. + \sum_{k=1}^{\infty} I_{\{\tau_k < t\}} (c_1 + c_2|X_{\tau_k^+} - X_{\tau_k}|)\right]\\
&= J(\tau, Y).
\end{aligned}$$

Step 2. Now let (τ^*, Y^*) be an impulse policy defined in (4.3)–(4.4), and denote by X^* the corresponding controlled inventory process. Then thanks to Proposition 3.4, all inequalities up to (4.6) in Step 1 are equalities:

$$\begin{aligned}
&F_* + \frac{1}{t}\limsup_{n\to\infty} \mathbb{E}_x[G(X^*_{t\wedge\sigma_n})]\\
&\quad = \frac{G(x)}{t} + \frac{1}{t}\mathbb{E}_x\left[\int_0^{t} c_0(X^*_s)\mathrm{d}s\right.\\
&\qquad \left. + \sum_{k=1}^{\infty} I_{\{\tau^*_k < t\}} c_1(X^*_{\tau^*_k}, X^*_{\tau^{*+}_k})\right].
\end{aligned} \tag{4.7}$$

We need to show that $\limsup_{t\to\infty} \limsup_{n\to\infty} t^{-1}\mathbb{E}_x[G(X^*_{t\wedge\sigma_n})] = 0$. To this end, we notice that $X^*_{\tau^{*+}_j} = z_*$ and hence an application of (4.1) yields that

$$\begin{aligned}
&\mathbb{E}_x[g_0(X^*_{t\wedge\sigma_n})]\\
&= \mathbb{E}_x\left[\sum_{j=0}^{\infty} I_{\{\tau^*_j < t\wedge\sigma_n \leq \tau^*_{j+1}\}}\left[\int_{\tau^*_j}^{t} -c_0(X^*_s)\mathrm{d}s + g_0(z_*)\right]\right]\\
&\leq \mathbb{E}_x\left[\sum_{j=0}^{\infty} I_{\{\tau^*_j < t\wedge\sigma_n \leq \tau^*_{j+1}\}} g_0(z_*)\right] = g_0(z_*).
\end{aligned}$$

On the other hand, since $X^*_t \geq y_*$ for all $t \neq \tau^*_j$, $j = 1, 2, \ldots$, and g_0 is increasing, it follows that $\mathbb{E}_x[g_0(X^*_t)] \geq g_0(y_*)$ for all $t \neq \tau^*_j$, $j = 1, 2, \ldots$. But $X^*_{\tau^*_j}$ may be strictly less than y_* due an exponential jump. That is, it is possible that $X^*_{\tau^{*-}_j} \geq y_*$ but $X^*_{\tau^*_j} = X^*_{\tau^{*-}_j} - \xi_i < y_*$ for some i with ξ_i being exponentially distributed, at which point, an impulse control pushes the inventory back to level $X^*_{\tau^{*+}_j} = z_*$. Recall that we have shown in Lemma 3.1 that g_0 is increasing under condition (3.7). Thus we can deduce

$$\begin{aligned}
\mathbb{E}_x[g_0(X^*_{\tau^*_j})] &= \mathbb{E}_x[g_0(X^*_{\tau^{*-}_j} - \xi_i))] \geq \mathbb{E}_x[g_0(y_* - \xi_i)]\\
&= \int_0^{\infty} g_0(y_* - y)\rho e^{-\rho y}\mathrm{d}y =: K_1 > -\infty.
\end{aligned}$$

Therefore, for any $t > 0$ and $n \in \mathbb{N}$, we have

$$-\infty < K_1 \wedge g_0(y_*) \leq \mathbb{E}_x[g_0(X^*_{t\wedge\sigma_n})] \leq g_0(z_*) < \infty,$$

and hence

$$\limsup_{t\to\infty} \limsup_{n\to\infty} t^{-1}\mathbb{E}_x[g_0(X^*_{t\wedge\sigma_n})] = 0.$$

Using the same argument, we can show that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} t^{-1}\mathbb{E}_x[\psi(X^*_{t\wedge\sigma_n})] = 0.$$

Then it follows that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} t^{-1}\mathbb{E}_x[G(X^*_{t\wedge\sigma_n})] = 0.$$

Plugging this equality in (4.7) gives us the desired conclusion $F_* = J(\tau^*, Y^*)$ and hence completes the proof. □

Remark 4.3. Note that the function G given in (3.14) is not bounded below. Thus it is not clear whether the transversality condition (4.2) will be satisfied for any admissible impulse control policy (τ, Y). However, for the policies (τ, Y) satisfying (4.2), Theorem 4.2 says that F_* is a lower bound for $J(\tau, Y)$ and that the (y_*, z_*)-policy defined in (4.3)-(4.4) achieves such a lower bound. In other words, Theorem 4.2 establishes the optimality of the (y_*, z_*)-policy in the restricted class of policies.

5. Further Remarks

This paper formulated and solved a long-term average inventory control problem where inventory dynamics is subject to diffusion and compound Poisson demands. Our approach extends the methodology previously developed in Helmes et al. (2017) for the corresponding problem in the continuous inventory setting, adapting it to account for the presence of compound Poisson jumps.

Several directions for further research are worth exploring. One important aspect is addressing the limitation of Theorem 4.2 imposed by the transversality condition (4.2). For instance, one may extend the comparison theorem for continuous inventory setting of He et al. (2017) to systems with jumps, thereby establishing the optimality of the (y_*, z_*)-policy for all admissible impulse control policies. Alternatively, a weak convergence approach similar to that in Helmes et al. (2018) may bypass (4.2).

Extending the model to incorporate more general inventory dynamics could yield valuable insights. For instance, using spectrally negative Lévy processes or jump-diffusion models may better capture real-world inventory fluctuations—particularly in industries with heavy-tailed demand spikes or non-exponential behavior. A further generalization to regime-switching (jump) diffusion processes (Mao and Yuan (2006), Yin and Zhu (2010)) would enable regime-dependent control policies, making the model even more adaptable to diverse stochastic environments. These extensions would not only advance the theoretical framework of stochastic impulse control but also significantly broaden its applicability to real-world problems.

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