The Need for Non-Gaussian Noise in Control System Models. Why Non-Gaussian Noise Matters?

Paweł D. Domański[1], Tyrone Duncan[2] and Bozenna Pasik-Duncan[2]

Abstract— **The noise in control systems was studied based on data from several hundreds of control loops operating in different process industries located in several sites all over the world. That data showed that the theoretical assumption of Gaussian properties for the data is hardly ever satisfied. This paper will focus on some illustrative examples of stochastic models with non-Gaussian noise and will present the evolution process in using stochastic processes that include fractional Brownian motion processes, Rosenblatt and Rosenblatt–Volterra processes as a replacement of commonly used ordinary Brownian motions. Theoretical advancements will demonstrate challenges and fascinating opportunities in them for developing the models that meet the expectations of industrial practitioners.**

Index Terms— **control system; uncertainty modeling; non-Gaussian noises; Rosenblatt process; Brownian motion**

I. Introduction

Virtually, no control system is completely insulated from the environment and external influences. These influences might be recognized, interpreted and taken into account in many different ways. Actually, this issue is still underestimated without being sufficiently understood whether in research or education. Historically in 1921, F. Knight presented the theory of uncertainties [1]. He pointed out that there exits the thin line separating two notions: "uncertainty" and "risk". He indicates the two types of uncertainty around us. The first one we can measure and than we call it "risk", while the second one cannot be determined (measured) and is names as a true "uncertainty", which is "pure and untainted". Uncertainties always exist, though often remain unnoticeable. If we are seriously impacted by them, we are forced to take them into account. If their impact is minor on undetected, we just ignore them. However, we mustn't confuse disturbance with uncertainty, as the consequence can be devastating.

Once the disturbance has known distribution, we may *a priori* know it, analyze and evaluate the associated risk. While it's statistical, it has an unknown distribution, but we can estimate it. In the worst case, the uncertainty is unclassified, has no distribution and therefore we may only observe its effects. From the research point of view the second case seems to be the most attractive.

The design of a given control system has to consider the disturbance investigation, and their active consideration. We must assess their impact on the achievable performance. The very fact of identifying their existence is a success. Once we are able to estimate their properties, we can propose an appropriate robust and resilient solution, like filtering, decoupling, or rejection. Generally, we observe two kinds uncertainties in control systems.

We can consider some of them as a kind of interference of unknown properties impacting the process, the most often of an external origin. We don't know when and how they occur, in what direction they change and with what magnitude. Most often we classify them as disturbances. However, they also exist in a context of noises. Noise is often considered as an internal phenomenon. We can model it as a stochastic process, like a thermal effects in sensor, "added" to the true information. The worst situation we may face is when we cannot distinguish true signal from the noise. Once the noise is known, we may take it into account, and properly design an adequate solution to mitigate its impact. Thus, the knowledge and investigation of noises matters a lot.

Noises in control system are modeled as the Gaussian normal process. It's well established tradition starting from Wiener deliberations on so called fire control during World War 2 [2], and widespread with works of Kalman on optimal control [3] and filtering [4]. The assumption about noise Gaussian properties is justified by its theoretical clarity and analytical vulnerability. Gaussian process has clear and compact formulation with all moments existing. Its incorporation in theoretical control system evaluations allows to use simple analytical solutions. The reason is also due to the Central Limit Theorem and an understanding that measurement noises exhibit normal properties.

Despite all the tradition and existing solutions developed using this assumption, analysis of actual industrial control systems does not support such an assumption. Reviews of large industrial datasets shows that only small number, $\approx 6\%$ of the loops meet normal properties [5]. Majority of signals exhibits heavy tails, which cannot be properly modeled by Gaussian probabilistic density function.

Incorrect choice of the noise underlying stochastic process may lead to wrong, mostly to the too optimistic interpretations, as in case of the estimation of benefits from control rehabilitation [6]. It's due to the fact that normal moments estimators exhibit 0% breakdown point [7] due to outlying observations hidden in heavy tails. Second moment for such data overestimates its value, what causes wrong decisions.

Existence of outliers and heavy tails requires stochastic tail-aware solutions [8] or alternative measures [9]. One approach is to use robust statistics [10], which allows to model the peak and shoulders of the distribution neglecting tails.

*This work was not supported by any organization

[1]Warsaw University of Technology, Institute of Control and Computation Engineering, Nowowiejska 15/19, 00-665 Warsaw, Poland pawel.domanski@pw.edu.pl

[2]University of Kansas, Department of Mathematics, 1460 Jayhawk Blvd., Lawrence, 660 45, Kansas, USA duncan@ku.edu; bozenna@ku.ed

Another approach is to use heavy tailed distributions as an underlying noise model. The family of α-stable distributions seems to be a natural selection [11], as normal function is its special case. Stable functions have four parameters, i.e. the shift, scale and two shape factors, which allow to model the broad range of skewed and tailed distributions. One can use simple, t-Student distribution [12], which effectively allows to model tailedness.

Recent research shows that Rosenblatt process [13] may be effectively applied to model noises in control engineering applications [14]. This paper investigates these opportunities.

II. How to capture control noise?

Modeling of the industrial control noise is not such simple as it appears in textbooks. First of all, we have to find it and this task is not easy. The noise is hidden in time series of control variables. The most often we look after it in the controlled (process) variable or in control error. These signals are composed of many elements, like static trends (for instance due to setpoint changes), artificial manipulations due to human interventions, oscillatory components of various frequencies and amplitudes, system or sensor based artifacts (external outliers), and finally the noise itself. Decomposition of time series to get the real noise is tedious and custom task, dependent on expertise and process knowledge. During these activities we shouldn't make any unjustified assumptions as they may alter the results. Especially, we mustn't assume any specific noise character.

Once the signal is decomposed and we obtain a component which we believe is noise, we can model it. At first we should make its visual inspection. Fig. 1 presents time series trend of sample noise signal. The review should be followed by the stationarity testing For that we may use a combination of an augmented Dickey–Fuller test (ADF), which tests the null hypothesis that a unit root is present in data sample [15] and Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test to check if data is trend-stationary or has a unit root [16]. Once time series is trend-stationary we may consider its differencing before further analysis. Next we should start the analysis of it statistical properties. Good practice is to start from histogram drawing. Even its simple visual review may deliver valuable information. Fig. 2 shows respective histogram plot. Histogram analysis should be supported by the evaluation of basic statistical factors, like its minimum and maximum value, mean μ and median, Q1 and Q3 quartiles and a boxplot (see Fig. 1).

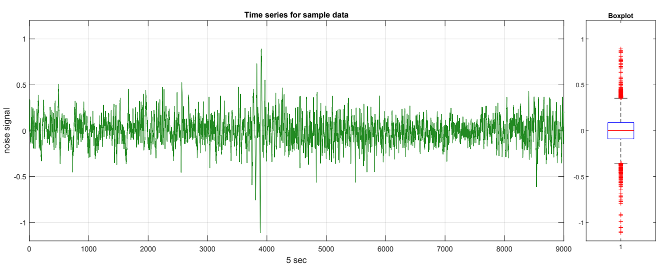

Fig. 1. Time series trend and the boxplot for sample data

Basic statistics might be supported by classical moments, i.e. standard deviation σ, skewness γ_3 and kurtosis γ_4. Additionally good practice suggest to evaluate MADAM (Median Absolute Deviation Around Median) and IQR (InterQuartile Range) scale factors. They might be supported by robust mean $\mu_{\log}$ and scale estimators $\sigma_{\log}$ (in this example logistic M-estimators) and L-moments [17]: L-scale l_2, L-covariance τ_2, L-skewness τ_3 and L-kurtosis τ_4. Table I presents these factors calculated for sample data.

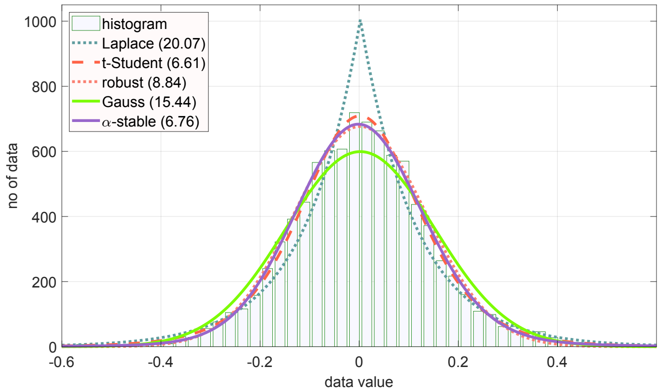

Fig. 2. Histogram plot for sample data

TABLE I
Sample data - basic statistics

min	max	μ	median	Q1	Q3
-1.111	0.891	0.002	0.002	-0.088	0.089
σ	MADAM	IQR	γ_3	γ_4	
0.150	0.088	0.087	-0.078	6.636	
l_2	τ_2	τ_3	τ_4	$\mu_{\log}$	$\sigma_{\log}$
0.081	34.653	0.008	0.176	0.002	0.133

Histogram plot might be used to select a probabilistic density function (PDF) that fits the data in the best way. Fig. 2 shows such fitting using the most appropriate candidates: normal Gaussian and its robust counterpart, double exponential Laplace, heavy tailed t-Student, which incorporates one shape factor and α-stable function that uses two shape coefficients: stability factor α representing tails and skewness β. Histograms allow to see fitting efficiency in each sector: peak, shoulders or tails. Fitting may be also visualized using quantile Q-Q plots as shown in Fig. 3. The plot may be also used to calculate fitting efficiency and to select the best function. In this case we see that α-stable function exhibits the best fitting and might be used as a noise model.

At that point we may conclude that the noise might be modeled by the random number generator using properly selected distribution. However, such a modeling looses one feature: the time dependence and causality relationship. The PDF is static. Each generated value is independent. Real noise might be like that. But generally, it can be generated by some unknown fundamental process, which output exhibits certain probabilistic properties. Therefore, it would be good to have possibility to choose between static, independent realizations and time driven fundamental process. Once the

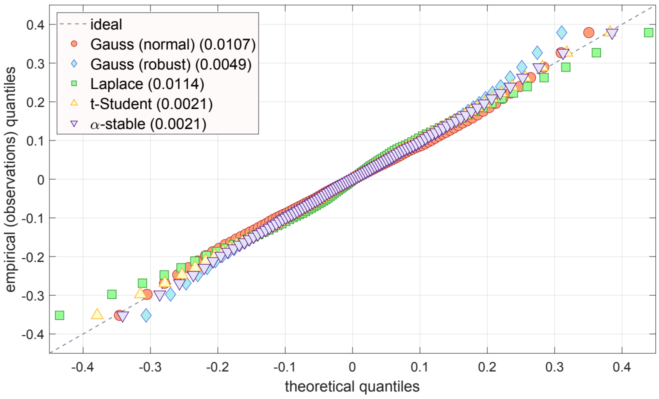

Fig. 3. Quantile Q-Q plot for sample data

process hypothesis is in place the modeling should use the statistical results leading toward the determination of proper fundamental process, like for instance fractional Brownian motion, Rosenblatt or Rosenblatt - Volterra processes.

A. *Comments*

The model of Brownian motion is typically justified by a Central Limit Theorem and an assumption of independence of the noise increments. However, typically data are not studied to justify either of these conclusions from real data. There is a wholesale acceptance of the naturality of a Gaussian assumption for the variation of an arbitrary collection of empirical data. Even more so no justification is given in a fundamental sense for the independent increments of the data that is basic to a Brownian motion model. In this paper an empirical analysis was used from physical data from interconnected control systems throughout the world. This analysis showed that the Gaussian assumption is hardly ever satisfied.

Thus, it is necessary to consider other noise models. In this paper an alternative noise model is chosen that is called Rosenblatt processes named after the statistician who introduced them [18]. There is a stochastic calculus associated with this family of processes which makes it feasible for applications to model physical system noise especially those cases where Brownian motion has traditionally been used. Some control system applications of optimal control are briefly described. Some references are provided for the foundations to control applications.

III. Rosenblatt Processes

To introduce Rosenblatt processes precisely, let $(u)_+ = \max\{u, 0\}$ be the positive part of u and h_1^H and h_2^H be given as

$$h_1^H(u, y) = (u - y)_+^{H-\frac{3}{2}} \tag{1}$$

$$h_2^H(u, y_1, y_2) = (u - y_1)_+^{\frac{H}{2}-1}(u - y_2)_+^{\frac{H}{2}-1} \tag{2}$$

that are two singular kernels defined on the real line. These two kernels are used to define fractional Brownian motions and Rosenblatt processes as follows.

Definition III.1. Let $H \in (1/2, 1)$ be fixed. A real-valued *fractional Brownian motion*, $B_H = (B_H(t), t \in \mathbb{R})$, is defined as follows

$$B_H(t) = C_H^B \int_{\mathbb{R}} \left(\int_0^t h_1^H(u, y) du \right) dW(y) \tag{3}$$

for $t \geq 0$ (and similarly for $t < 0$) where C_H^B is a constant given below such that $\mathbb{E}(B_H^2(1)) = 1$ and W is a standard Wiener process (Brownian motion) on a fixed complete probability space denoted $(\Omega, \mathcal{F}, \mathbb{P})$ which is used throughout this paper.

Definition III.2. Let $H \in (1/2, 1)$ be fixed. A real-valued (standard) *Rosenblatt process*, $R_H = (R_H(t), t \in \mathbb{R})$, is defined as follows

$$R_H(t) = C_H^R \int_{\mathbb{R}^2} \left(\int_0^t h_2^H(u, y_1, y_2) du \right) dW(y_1) dW(y_2)$$

for $t \geq 0$ (and similarly for $t < 0$) where C_H^R is a constant such that $\mathbb{E}(R_H^2(1)) = 1$, and the double stochastic integral is a Wiener-Itô multiple integral of order two with respect to the Wiener process (standard Brownian motion) W. This double Wiener integral is the definition given by Itô so the integral has expectation zero [19] in contrast to Wiener's original definition.

A. *A Change of Variables Formula*

Anyone who has developed models for control and filtering knows the importance of a change of variables formula, often called the Itô formula. Similarly a change of variables formula is basic for the analysis of systems with a Rosenblatt noise process. This change of variables result is described now.

A change of variables formula is important in this case as it is for a Brownian motion noise control problem. To describe a change of variables formula let $(y(t), t \geq 0)$ be a real-valued stochastic process that satisfies the following stochastic differential equation where $y(0) = y_0$ and R_H is a Rosenblatt process with parameter H.

$$dy(t) = \vartheta(t)dt + dR_H(t)$$

If some natural conditions are satisfied for every $T > 0$, then the process $(Y_t)_{t \geq 0}$ defined by $Y_t = f(t, y_t)$ satisfies the following stochastic equation that is verified in [13].

$$Y_t = Y_0 + \int_0^t \tilde{\vartheta}_s ds + 2c_H^{B,R} \int_0^t \tilde{\varphi}_s dB_s^{\frac{H}{2}+\frac{1}{2}} + \int_0^t \tilde{\psi}_s dR_s^H$$

for every $t \geq 0$ where

$$\begin{aligned} \tilde{\vartheta}_s &= \frac{\partial f}{\partial s}(s, y_s) + \frac{\partial f}{\partial x}(s, y_s)\vartheta_s \\ &\quad + c_H^R \frac{\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2}, \frac{H}{2}} y_s)(s, s) \\ &\quad + c_H^R \frac{\partial^3 f}{\partial x^3}(s, y_s)[(\nabla^{\frac{H}{2}} y_s)(s)]^2, \\ \tilde{\varphi}_s &= \frac{\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2}} y_s)(s), \\ \tilde{\psi}_s &= \frac{\partial f}{\partial x}(s, y_s). \end{aligned}$$

B. Linear prediction problem

Now some linear prediction problems for Rosenblatt processes are formulated and solved. This approach for the linear prediction problem was suggested by one of the referees. Let h be a positive continuous function on the real line. Find the optimal linear predictor of $\int_0^t h dR_H$ based on the past $(\int_u^v h dR_H, -s < u < v \leq 0)$. There is the following result.

Theorem III.3. *Fix $t > 0$. The optimal linear prediction of the random variable $\int_0^t h dR_H$ for h given above based on the history $(\int_u^v h dR_H, -s < u < v \leq 0)$ for some $s > 0$ is given by*

$$\tilde{R}_H(s) = \int_{-s}^{0} \hat{g}(x) dR_H(x) \tag{4}$$

where

$$\widehat{g}_1(x) = \frac{1}{\pi} \sin\left(\pi\left(H - \frac{1}{2}\right)\right) |x|^{\frac{1}{2}-H}(s+x)^{\frac{1}{2}-H} \int_0^t \frac{y^{H-\frac{1}{2}}(y+s)^{H-\frac{1}{2}}}{y+|x|} \mathrm{d}y, \ x \in (-s, 0). \tag{5}$$

Proof. Recall that the optimal estimator arises from the orthogonality property

$$\mathbb{E}\left(\int_0^t h(r)\mathrm{d}R_H(r) - \int_{-s}^{0} \widehat{g}(x)\mathrm{d}\left(\int_{-s}^{\bullet} h\mathrm{d}R_H\right)(x)\right) \left(\int_u^v h(r)\mathrm{d}R_H(r)\right) = 0. \tag{6}$$

It is basic to determine the optimal linear estimator to find the function $\hat{g}$ where

$$\widehat{g}(x) := G\left(\frac{-x}{s}\right)(h(x))^{-1}, \quad x \in (-s, 0) \tag{7}$$

and

$$G(x) := -c(H)^{-1} x^{\frac{1}{2}-H} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^1 \left(\xi^{2H-1}(\xi - x)^{\frac{1}{2}-H} \frac{\mathrm{d}}{\mathrm{d}\xi} \int_0^{\xi} \eta^{\frac{1}{2}-H}(\xi-\eta)^{\frac{1}{2}-H} F(\eta)\mathrm{d}\eta\right) \mathrm{d}\xi \tag{8}$$

with the constant $c(H)$ defined by

$$c(H) := 2\cos\left(\frac{1}{2}\pi(1-2H)\right)\Gamma(2H-1)\Gamma\left(\frac{3}{2} - H\right)^2 \tag{9}$$

and with the function F defined by

$$F(y) := s^{1-2H} \int_0^t h(x)(x+sy)^{2H-2}\mathrm{d}x, \quad y \in (0, 1) \tag{10}$$

and

$$G(x) = \hat{g}(-sx)h(-sx), \qquad x \in (0, 1) \tag{11}$$

The expression here for the optimal linear estimate is analogous to the prediction result for a fractional Brownian motion.

□

For a scalar constant coefficient linear differential equation given by

$$dX(t) = aX(t)dt + dR_H(t) \tag{12}$$

the solution is

$$X_t = X_0 + \int_0^t \mathrm{e}^{a(t-r)}\mathrm{d}R_H(r), \quad t > 0, \tag{13}$$

One has an analogous solution for the optimal linear estimator given the observations $(X(u), u \in (-s, 0))$ as follows

$$\widehat{X}_t = \mathrm{e}^{at}X_0 + \mathrm{e}^{at}\int_{-s}^{0} \widehat{g}_{exp}(r)\mathrm{d}(\mathrm{e}^{-a\bullet}X_\bullet)(r) \tag{14}$$

where

$$\widehat{g}_{exp}(x) := \mathrm{e}^{ax}G_{exp}(-x/s), \quad x \in (-s, 0) \tag{15}$$

$$G_{exp}(x) = -c(H)^{-1}x^{\frac{1}{2}-H}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \left(\xi^{2H-1}(\xi-x)^{\frac{1}{2}-H}\frac{\mathrm{d}}{\mathrm{d}\xi} \int_0^{\xi} \eta^{\frac{1}{2}-H}(\xi-\eta)^{\frac{1}{2}-H}F_{exp}(\eta)\mathrm{d}\eta\right)\mathrm{d}\xi \tag{16}$$

The term $c(H)$ is given above and

$$F_{exp}(y) = s^{1-2H}\int_0^t \mathrm{e}^{-ax}(x+sy)^{2H-2}\mathrm{d}x, \quad y \in (0, 1). \tag{17}$$

This result reduces to the linear prediction for the Rosenblatt process.

A prediction problem solution is given for a Rosenblatt process. Recall that R_H denotes the Rosenblatt process in an interval on the real line. A well known problem for many stochastic processes is prediction. Here a result is given for the prediction of a Rosenblatt process that is described in the following theorem.

Theorem III.4. *Fix $t > 0$. The optimal linear predictor of $R_H(t)$ given $(R_H(s), u < s < 0)$, is*

$$\hat{R}_H(s) = \int_2^0 \hat{g}_1 dR_H \tag{18}$$

where

$$\hat{g}_1(x) = \frac{1}{\pi}sin(\pi(H - \frac{1}{2}))|x|^{\frac{1}{2}-H}(s+x))^{\frac{1}{2}-H} \int_0^t \frac{y^{H-\frac{1}{2}}(y+s)^{H-\frac{1}{2}}}{y+|x|}dy, \tag{19}$$

with $x \in (-s, 0)$.

This result is verified elsewhere.

IV. Conclusion

The modeling of industrial disturbances remains an orphan, or forgotten issue. We make some assumptions in control engineering, which are suitable for our analysis, but are hardly justified by an industrial practice. Observation of the literature simulations shows that process disturbances are generally modeled as steps, or a periodic rectangle or sinusoidal wave, added to the controller output before the process. Measurement noises are often added to the process variable (process output) and are almost always modeled with a normal Gaussian noise $\mathcal{N}(0, \sigma^2)$ with zero mean value and variance σ^2.

In majority, the research does not go beyond the above. A review of real data reveals a much greater richness of disturbances. In addition to step changes or oscillations, in most cases we are dealing with non-Gaussian stochastic processes. We observe definitely non-Gaussian behavior, often asymmetric with long or heavy tails.

Disturbances are very often not independent processes, indicating persistent or non-persistent properties. Long-range memory in signals is no exception and nonlinearities are more common rather than rarer. It is also worth remembering that process disturbances are very often non-stationary, if only because of changes in operating points.

Thus, to be able to design control systems well, we should not only concern ourselves with modeling the actual processes, but also the disturbances that affect it. Only then will our model of reality become complete, and the reliability of our control system design and the resulting strategy itself will be fully adequate.

This paper takes a first step in this direction, proposing the Rosenblatt process as a potential stochastic model to better represent industrial reality.

References

[1] F. Knight, *Risk, Uncertainty, and Profit*. Boston MA: Hart, Schaffner and Marx, Houghton Mifflin, 1921.

[2] N. Wiener, "Statistical method of prediction in fire control," Office of Scientific Research and Development, Washington, DC, Tech. Rep. ADA800106, Dec. 1942.

[3] R. Kalman, "Contributions to the theory of optimal control," *Boletin de la Sociedad Matematica mexicana*, vol. 5, pp. 102–119, 1960.

[4] ——, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering*, vol. 82, no. 1, pp. 35–45, 03 1960.

[5] P. Domański, "Non-gaussian properties of the real industrial control error in SISO loops," in *Proceedings of the 19th International Conference on System Theory, Control and Computing*, 2015, pp. 877–882.

[6] ——, "Non-Gaussian assessment of the benefits from improved control," in *Preprints of the IFAC World Congress 2017*, Toulouse, France, 2017, pp. 5092–5097.

[7] P. Rousseeuw and A. Leroy, *Robust Regression and Outlier Detection*. New York, NY, USA: John Wiley & Sons, Inc., 1987.

[8] P. D. Domański, *Back to Statistics. Tail-Aware Control Performance Assessment*. Boca Raton, FL: CRC Press, Taylor & Francis Group, 2025.

[9] P. Domański, "Control quality assessment using fractal persistence measures," *ISA Transactions*, vol. 90, pp. 226–234, 2019.

[10] P. Huber and E. Ronchetti, *Robust Statistics, 2nd Edition*. Wiley, 2009.

[11] C. Nikias and M. Shao, *Signal processing with alpha-stable distributions and applications*. New York, NY: Wiley-Interscience, 1995.

[12] A. Shah, A. Wilson, and Z. Ghahramani, "Student-t Processes as Alternatives to Gaussian Processes," in *Proceedings of the Seventeenth International Conference on Artificial Intelligence and Statistics*, ser. Proceedings of Machine Learning Research, J. Kaski, S.and Corander, Ed. Reykjavik, Iceland: PMLR, 2014, vol. 33, pp. 877–885.

[13] P. Čoupek, T. Duncan, and B. Pasik-Duncan, "A stochastic calculus for Rosenblatt processes," *Stochastic Processes and their Applications*, vol. 150, pp. 853–885, 2022.

[14] B. Pasik-Duncan and T. Duncan, "On advances in noise modeling in stochastic systems, control and adaptive control," in *Adaptive Control Theory and Applications*, P. Ioannou, Ed. Rijeka: IntechOpen, 2024, ch. 2.

[15] D. Dickey and W. A. Fuller, "Distribution of the estimators for autoregressive time series with a unit root," *Journal of the American Statistical Association*, vol. 74, no. 366, pp. 427–431, 1979.

[16] D. Kwiatkowski, P. Phillips, P. Schmidt, and S. Y., "Testing the null hypothesis of stationarity against the alternative of a unit root," *Journal of Econometrics*, vol. 54, no. 1–3, pp. 159–178, 1992.

[17] J. Hosking, "L-moments: Analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 52, no. 1, pp. 105–124, 1990.

[18] M. Taqqu, "Weak convergence to fractional brownian motion and to the rosenblatt process," *Zeitschrift fűr Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 31, pp. 287–302, 1975.

[19] P. Čoupek, T. Duncan, B. Maslowski, and B. Pasik-Duncan, "An infinite time horizon linear-quadratic control problem with a Rosenblatt process," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 4973–4977.