Bounded Confidence Opinion Dynamics in Non-Euclidean Norms: Containment and Convergence with Stubborn Agents

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***Abstract*— Bounded confidence models of opinion formation are paradigmatic examples of coevolutionary networks, where agents update their opinions by averaging those of peers deemed sufficiently similar. In this paper, we study a multidimensional extension of the Hegselmann–Krause (HK) model with stubborn agents (MHK-S), in which the confidence sets are defined as balls in a general (not necessarily Euclidean) norm and some agents remain stubborn, being resistant to social influence. We establish a behavioral dichotomy: an agent's opinion either terminates in finite time (if not influenced by stubborn agents) or converges asymptotically to the convex hull of the stubborn agents' opinions. In the special case where the stubborn agents' opinions are sufficiently close, the influenced agents converge to their barycenter; we further extend this result to scenarios featuring multiple clusters of stubborn opinions that are sufficiently distant from each other.**

I. INTRODUCTION

Bounded confidence models of opinion formation are paradigmatic examples of coevolutionary networks, where the graph structure and node states evolve interdependently. Each node represents an individual social agent with a real-valued opinion. At each step, every agent has access to all opinions in the group, yet she selects only those that are sufficiently similar to her own, dismissing the dissimilar ones; the resulting influence graph is inherently opinion-dependent. Subsequently, agents update their opinions by adopting the average of the selected opinions. In the Hegselmann-Krause model (**HK**) [1]–[3], being the primary focus of this paper, all opinions are updated synchronously, whereas the Deffuant-Weisbuch model [4] employs asynchronous gossip-based updates. Both models, renowned for their highly complex nonlinear dynamics, have attracted significant attention from a broad segment of the research community.

In the classical HK model, opinions are scalars. These models have been extensively studied, and explicit estimates for convergence time have been derived; see [5], [6] for a survey of recent results and discussion of open problems. Although studies on scalar opinions dominate the literature, many properties extend to multidimensional Hegselmann-Krause dynamics, where confidence intervals are generalized to balls defined by a norm on $\mathbb{R}^n$ [7]–[9] or to other sets [10], [11]. An analysis of the literature on the multidimensional HK model reveals two significant limitations.

The first limitation is the reliance on the Euclidean norm to measure opinion distances. Although asymptotic convergence of opinions—assuming equal confidence radii—can be established in any norm [6] and primarily relies on the convergence of type-symmetric matrix products [12], all known Lyapunov-based results are confined to Euclidean norms. The Euclidean norm guarantees the existence of pseudo-quadratic Lyapunov functions [7], [9], which allow for an elegant graph-theoretic interpretation [13], [14]. At the same time, there is no compelling experimental evidence to support the exclusive use of the Euclidean norm (or any particular norm) for measuring the proximity of opinions within the cognitive mechanisms of social homophily. As argued in [15], the Manhattan (ℓ_1) norm is just as valid as the ℓ_2 norm. Recent data science studies [16] suggest that the ℓ_1 norm is preferable for comparing relative distances in high-dimensional random datasets, as it offers improved "contrast"–a higher ratio of maximal to minimal distance.

The second limitation, which persists even in studies of the scalar bounded confidence model, is the assumption of homogeneity in the agents' confidence radii. Without this assumption, even the asymptotic convergence of opinions remains an open question[1]. Some important breakthroughs, however, have been made in the special case where agents are divided into two distinct groups: fully stubborn agents, who maintain fixed opinions, and agents following the standard Hegselmann-Krause model, who update their opinions by averaging those of sufficiently similar peers—including the stubborn ones. In the case where the stubborn agents share a common opinion, the convergence can be proven by using the s-energy (power series) method [18] or the theory of recurrent matrix inequalities [11]. Except for the Euclidean norm case, where convergence has been recently proven in [19], the scenario of heterogeneous stubborn agents' opinions remains largely unexplored[2].

Here, we address this broader case and present the following results. **First,** we establish a behavioral dichotomy: an agent's opinion either stabilizes in finite time—if, from some point onward, she is not directly or indirectly influenced by stubborn individuals—or converges asymptotically to the

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[1]Notice that convergence under heterogeneous confidence radii has been recently proved for the Deffuant-Weisbuch model [17].

[2]Notice that replacing the norms by more general confidence mechanisms with non-convex confidence sets [11], opinions can oscillate in the presence of two distinct stubborn agents.

convex hull of the stubborn agents' opinions. Note that this convergence is asymptotic rather than finite-time. In the multi-agent control literature, such behavior is often referred to as containment [20]. **Second,** if the opinions of the stubborn individuals are sufficiently close, then the opinions of all agents influenced by them eventually converge to the barycenter of the stubborn individuals' opinions. **Third,** we extend this result to a more general scenario involving multiple clusters of closely aligned stubborn opinions, provided that the distances between clusters are sufficiently large. In this scenario, each opinion either stabilizes in finite time or converges to the barycenter of one of the clusters.

II. The Model at Hand

The bounded confidence model we analyze corresponds to the multidimensional HK model introduced in [19], in which the "inertia" coefficients of all agents are either 0 (for *regular* agents) or 1 (for *stubborn* agents); however, the norm in the opinion space $\mathbb{R}^d$ need not be Euclidean.

Agents. Let $\mathcal{V}$ denote the set of agents and let $n = |\mathcal{V}|$ be their total number[3]. Among these agents, we distinguish between *regular* and *stubborn* agents, denoted by the sets $\mathcal{V}_r$ and $\mathcal{V}_s$, respectively, so that $\mathcal{V} = \mathcal{V}_r \cup \mathcal{V}_s$.

Opinions. At time step $t = 0, 1, \ldots$, agent $i \in \mathcal{V}$ holds a multidimensional opinion $\xi^i(t) \in \mathbb{R}^d$, where each element ξ^i_k represents the agent's position on topic $k \in 1, \ldots, d$. The system's state can be written as matrix $\Xi \overset{\Delta}{=} (\xi^i_k) \in \mathbb{R}^{n\times d}$.

Confidence graph. Each regular agent forms their opinions based on the similar opinions of her peers, with "similarity" defined by proximity in some norm $\|\cdot\|$ on $\mathbb{R}^d$, which need not be Euclidean (i.e., to be induced by some inner product). It is convenient to introduce a *confidence graph* $\mathcal{G}(\Xi) = (\mathcal{V}, \mathcal{E}(\Xi))$. In this graph, the nodes represent the agents, and node $i \in \mathcal{V}$ has the set of (out-)neighbors

$$\mathcal{N}_i(\Xi) \overset{\Delta}{=} \{j \in \mathcal{V} : \|\xi^j - \xi^i\| \le R\}. \tag{1}$$

In other words, an arc $i \to j$ exists if and only if $\|\xi^j - \xi^i\| \le R$, where R is the *confidence radius*.

Opinion Update Rules. The opinions of stubborn agents (called also radicals or zealots) remain constant:

$$\xi^i \equiv \xi^i(0) \quad \forall i \in \mathcal{V}_s. \tag{2}$$

The regular agents update their opinions by averaging the opinions they have confidence in, following the standard rule

$$\xi^i(t+1) = \frac{1}{|\mathcal{N}_i(\Xi(t))|} \sum_{j\in\mathcal{N}_i(\Xi(t))} \xi^j(t), \quad i \in \mathcal{V}_r. \tag{3}$$

Henceforth the opinion dynamics (2), (3) is referred to as the MHK-S, standing for "Multi-Dimensional Hegselmann-Krause with Stubborn agents". We denote by $\mathcal{P}$ the convex hull of the stubborn agents' opinions, i.e., the polyhedron

$$\mathcal{P} \overset{\Delta}{=} \mathrm{conv}\{\xi^i\}_{i\in\mathcal{V}_s}. \tag{4}$$

[3]Hereinafter, the cardinality of a finite set N is denoted by $|N|$.

III. Behavioral Dichotomy

Our first result shows that all opinions in the MHK-S model fall into one of two categories: either their dynamics terminate in finite time, or they converge to the convex polytope $\mathcal{P}$ defined in (4), as a result of persistent influence by stubborn agents. We begin with the following definition.

Definition 1: Consider a specific trajectory of the MHK-S system, $\Xi(t)$. We say that along this trajectory, agent $i \in \mathcal{V}$ has *persistent confidence in* agent $j \in \mathcal{V}$ if $j \in \mathcal{N}_i(\Xi(t))$ occurs infinitely often. By connecting all such pairs (i, j), we construct the *persistent confidence graph*, $\mathcal{G}_\infty(\Xi(\cdot))$. A regular agent $i \in \mathcal{V}_r$ is said to be *S-influenced* if there exists a path in $\mathcal{G}_\infty(\Xi(\cdot))$ connecting i to at least one stubborn agent[4] $j \in \mathcal{V}_s$; otherwise, agent i is *S-independent*[5].

The following theorem is our first main result.

Theorem 1: For each solution of the MHK-S, the following two statements are true:

(i) The opinion of each S-influenced agent i converges to $\mathcal{P}$, i.e., $\mathrm{dist}(\xi^i(t), \mathcal{P}) = \min_{\eta\in\mathcal{P}} \|\xi^i(t) - \eta\| \xrightarrow[t\to\infty]{} 0$;

(ii) For every S-independent agent j, the opinion terminates in finite time at a stationary value $\bar{\xi}^j$, so that $\xi^j(t) \equiv \bar{\xi}^j$ for all sufficiently large t. For any two S-independent agents j and k, either $\bar{\xi}^j = \bar{\xi}^k$ or $|\bar{\xi}^j - \bar{\xi}^k| > R$. $\star$

A. A numerical example

Before presenting the proof, we illustrate Theorem 1 with a numerical example with $n = 100$ agents. In this example, the initial opinions $\xi^i(0) \in \mathbb{R}^2$ of 98 regular agents are uniformly sampled from the domain $[-1, 1] \times [-1, 1]$, while the opinions of two stubborn agents are fixed at $(0.2, -0.38)$ and $(-0.5, -0.86)$. The dynamics of MDHK-S with the confidence balls of radius $R = 0.5$, defined in ℓ_1-norm, during the first 40 time steps is illustrated in Fig. 1. In this

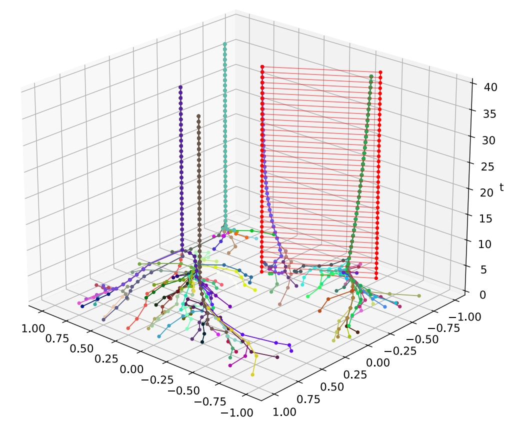

Fig. 1. MHK-S Dynamics with Two Stubborn Agents: Multiple Clusters

figure, the vertical axis represents time, while the horizontal

[4]This means that the influence of stubborn agents on an S-influenced agent—whether direct or indirect—may be interrupted for arbitrarily long periods but never ceases entirely.

[5]Notice that the sets of S-influenced and S-independent agents are trajectory-dependent, and predicting whether a given agent will be S-influenced or S-independent is a non-trivial problem.

plane corresponds to the two-dimensional opinions. The set $\mathcal{P}$ is a line segment connecting the two stubborn agents' opinions; to illustrate the time convergence of trajectories, we depict the plane $\mathcal{P} \times [0, \infty)$ as a red-shaded area.

In this experiment, the opinions of the S-independent agents coalesce into three distinct clusters[6] within the first 10 steps. The remaining agents form two clusters, each interacting exclusively with one stubborn agent and eventually converging to that agent's opinion. One may observe that the convergence rates of these two groups of S-influenced opinions differ noticeably, which is caused by their sizes – 7 agents in the "fast" cluster versus 29 agents in the "slow" cluster. Indeed, if m identical opinions, $\xi^1(t) = \ldots = \xi^m(t)$, are influenced by a constant stubborn opinion ξ^0 starting from time $t \geq t_0$, then these opinions obey the equation:

$$\xi^i(t+1) = \frac{m}{m+1}\xi^i(t) + \frac{1}{m}\xi^0,\ t \geq t_0.$$

Hence, $\|\xi^i(t) - \xi^0\| = \theta^{t-t_0}\|\xi^i(t_0) - \xi^0\|$, with $\theta = \frac{m}{m+1}$. This shows that the larger the cluster size m, the closer θ to 1 slower the convergence of opinions.

It should be noted that the opinions of S-influenced agents need not converge to one of the vertices of the polytope $\mathcal{P}$; an important case in which their final opinion lies strictly inside this set will be considered in the next section. Remarkably, even if an agent's initial opinion is very close to that of a stubborn agent, the agent may still be S-independent. This phenomenon is demonstrated in the second experiment, where the opinions of the stubborn individuals remain same as before, but the initial opinions of regular agents are different. In this scenario, the opinions of all agents reach consensus (and terminate) after 18 steps, and the final consensus opinion is distant from $\mathcal{P}$.

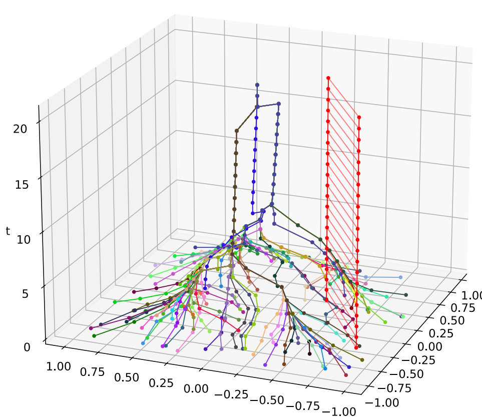

Fig. 2. MHK-S Dynamics with Two Stubborn Agents: Consensus

Note that Theorem 1 formally does not guarantee that the opinions of S-influenced agents converge to equilibria; however, we have not observed any oscillatory behavior in our experiments, regardless of the norm, radius R, or initial opinions chosen. We discuss this in more detail below.

[6]A cluster is a group of agents with identical opinions; clearly, once such a group is formed, the opinions of its members remain coincident.

B. Proofs

The proof of Theorem 1 generalizes the approach of [11] and leverages the theory of recurrent averaging inequalities (RAI). By definition, a RAI is an inequality of the form

$$x(t+1) \leq W(t)x(t), \quad t = 0, 1, \ldots, \tag{5}$$

where $x(t) \in \mathbb{R}^m$, $W(t)$ are row-stochastic $m \times m$ matrices, and the inequality is understood elementwise. The following lemma was established as Theorem 5 in [21].

Lemma 1: Let matrices $W(t)$ be *type-symmetric*, that is, for some constant $K \geq 1$ one has $K^{-1}w_{ji}(t) \leq w_{ij}(t) \leq K w_{ji}(t)$ for all pairs $i \neq j$ and all $t = 0, 1, \ldots$ Assume also that the diagonal entries are uniformly positive: $w_{ii}(t) \geq \delta > 0$ for all i and $t \geq 0$. Then, any feasible solution $x(t)$ of (5) that is bounded from below enjoys the following properties:

(a) a finite limit $x(\infty) \triangleq \lim_{t\to\infty} x(t)$ exists;
(b) $x_i(\infty) = x_j(\infty)$ for all pairs of agents i, j that interact *persistently*, that is, $\sum_{t=0}^{\infty} w_{ij}(t) = \infty$;
(c) the residuals $\Delta(t) \triangleq W(t)x(t) - x(t+1)$ are ℓ_1-summable, that is, $\sum_{t=0}^{\infty} \Delta_i(t) < \infty$ for all i. ⋆

Proof of Theorem 1. Notice first that, in the absence of stubborn agents ($\mathcal{V}_s = \emptyset$), Theorem 1 reduces to a well-known property of the homogeneous HK models. This property follows from the seminal Lorenz theorem on the convergence of type-symmetric matrix products [12] and can also be derived from Lemma 1 (see, e.g., the first part of the proof in [11] or [6, Theorem 13]). The next observation is that, starting from some sufficiently large time, the S-independent agents effectively follow a reduced (multidimensional) HK model without stubborn individuals. Indeed, denoting the set of S-independent agents by I and the remaining agents by $J = \mathcal{V} \setminus I$, then the persistent confidence graph $\mathcal{G}_\infty$ contains no arcs from nodes in I to nodes in J. That is, for sufficiently large t, we have $j \notin \mathcal{N}_i(t)$ for every pair with $i \in I$ and $j \in J$, meaning that the opinions $\{\xi^i(t)\}_{i\in I}$ for $i \in I$ evolve *independently* of the opinions $\{\xi^j(t)\}_{j\in J}$ and, in particular, are not influenced by the stubborn agents, because $\mathcal{V}_s \subseteq J$.

This proves the statement (ii) in Theorem 1, namely, that all opinions $\xi^i(t)$ terminate in finite time at some stationary values $\bar{\xi}^i$, $i \in I$ that are either coincident or distant:

$$\|\bar{\xi}^i - \bar{\xi}^k\| \in \{0\} \cup (R, \infty) \quad \forall i, k \in I. \tag{6}$$

We now proceed to prove statement (i). Consider a specific solution $\Xi(\cdot)$ to the MHK-S and denote $x_i(t)$ be the distance from agent i's opinion to the polytope $\mathcal{P}$, i.e.,

$$x_i(t) = \mathrm{dist}(\xi^i(t), \mathcal{P}) = \min_{\xi\in\mathcal{P}} \|\xi^i(t) - \xi\| \quad \forall i \in \mathcal{V}.$$

Since, starting from some time $t_0 \geq 0$, the S-independent agents hold static opinions that do not affect the S-influenced agents, we may, without loss of generality, assume that all agents are S-influenced. By reindexing the agents, we assume that $\mathcal{V}_r = \{1, \ldots, m\}$ and $\mathcal{V}_s = \{m+1, \ldots, n\}$. Trivially, $x_i(t) = 0$ for $i \in \mathcal{V}_s = \{m+1, \ldots, n\}$. The vector $x(t) = (x_1(t), \ldots, x_m(t))^\top \in \mathbb{R}^m$ (corresponding to the regular agents) is a feasible solution to the RAI (5), where the stochastic matrices $W(t) \in \mathbb{R}^m$ are as follows

$$w_{ij}(t) \triangleq \begin{cases} \frac{1}{|\mathcal{N}_i(t)|}, & i,j \in \mathcal{V}_r,\, i \neq j, j \in \mathcal{N}_i(t) \\ \frac{1+|\mathcal{N}_i(t)\cap\mathcal{V}_s|}{|\mathcal{N}_i(t)|}, & i = j \in \mathcal{V}_r, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Indeed, the dynamics (3) entail that, for each $i \in \mathcal{V}_r$,

$$\xi^i(t+1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{V}_r \cap \mathcal{N}_i(t)} \xi^j + \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{V}_s \cap \mathcal{N}_i(t)} \xi^j.$$

The distance function $\mathrm{dist}(\cdot, \mathcal{P}) : \mathbb{R}^d \to [0, \infty)$ is convex, since $\mathcal{P}$ is a convex set. Therefore,

$$\begin{aligned} x_i(t+1) &\leq \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{V}_r \cap \mathcal{N}_i(t)} x_j(t) \leq \\ &\leq \frac{1}{|\mathcal{N}_i(t)|} x_i(t) + \sum_{j \in \mathcal{V}_r \setminus \{i\}} w_{ij}(t) x_j(t) \leq \\ &\leq \sum_{j \in \mathcal{V}_r} w_{ij}(t) x_j(t) \quad \forall i \in \mathcal{V}_r. \end{aligned}$$

In view of the definition of $\mathcal{N}_i(t)$, the type symmetry condition holds with $K = n$. Consequently, there exists a limit $x^* = \lim_{t\to\infty} x(t)$, and the residual sequence $\Delta(t) = W(t)x(t) - x(t+1)$ is ℓ_1-summable (and, in particular, converges to 0). Moreover, we know that $x_i^* = x_j^*$ whenever agents i and j are connected in the graph $\mathcal{G}_\infty(\Xi(\cdot))$, because in this case, obviously, $\sum_{t=0}^\infty w_{ij}(t) = \infty$.

It is now straightforward to prove that $x^* = 0$. Indeed, if an agent $i \in \mathcal{V}_r$ interacts with at least one stubborn agent (i.e., $\mathcal{N}_i(t) \cap \mathcal{V}_s \neq \emptyset$) infinitely often, then it follows immediately that $x_i^* = 0$, as indicated by the inequality

$$\Delta_i(t) \geq x_i(t) \left(w_{ii}(t) - \frac{1}{|\mathcal{N}_i(t)|} \right) = \frac{|\mathcal{N}_i(t) \cap \mathcal{V}_s|}{|\mathcal{N}_i(t)|} x_i(t).$$

Furthermore, if an agent j repeatedly considers the opinion of such an agent i (i.e., $i \in \mathcal{N}_j(t)$ infinitely often), then we also have $x_j^* = 0$. In general, if there exists a path in the persistent confidence graph $\mathcal{G}_\infty$ connecting an agent j to $\mathcal{V}_s$, then $x_j^* = 0$. Since, by assumption, all agents are S-influenced, we conclude that $x^* = 0$, meaning that the opinions of S-influenced agents converge to the polyhedron $\mathcal{P}$. This completes the proof of statement (i).

C. Remarks and Discussions

Generalizations. Analyzing the proof of Theorem 1 reveals that it can be generalized in various directions. For example, following [19], each pair of agents could be assigned its own confidence radius $R_{ij} = R_{ji} > 0$. Furthermore, the dynamics of agents can be generalized by introducing inertia coefficients $h_i \in [0, 1]$ as in [19]:

$$\xi^i(t+1) = (1-h_i)\xi^i(t) + \frac{h_i}{|\mathcal{N}_i(\Xi(t))|} \sum_{j \in \mathcal{N}_i(\Xi(t))} \xi^j(t),\ i \in \mathcal{V},$$

where $h_i \in [0, 1]$. Clearly, $h_i = 0$ corresponds to stubborn agents, whereas agents with $h_i = 1$ follow the usual HK opinion update rule. In the inertia model, however, the opinions of S-independent agents do not terminate but instead exhibit asymptotic convergence.

Another possible generalization of Theorem 1 is concerned with the set-based confidence opinion dynamics (SCOD) introduced in our recent work [11]. Notice that the norm in (6) need not be the same as the one used in defining the confidence sets. Moreover, these sets need not be balls, and the definition of $\mathcal{N}_i$ can be modified as follows:

$$\mathcal{N}_i(\Xi) = \{j \in \mathcal{V} : \xi^j - \xi^i \in \mathcal{O}\}.$$

Here, $\mathcal{O}$ is a symmetric set (i.e., $\mathcal{O} = -\mathcal{O}$) whose interior contains the zero vector [11]. An example, for instance, is the ball in ℓ_p metric, which is non-convex for $0 < p < 1$.

Containment control. While the result of Theorem 1 is reminiscent of properties in containment control algorithms[7], it is not commonly found in the literature. Containment control, primarily studied in the context of mobile robotics, has been extensively explored in continuous time [20], [24]. In contrast, discrete-time containment control algorithms are less developed and often impose restrictions on time-varying graphs that are challenging to verify for the state-dependent Hegselmann-Krause model [25], [26]. We thus give a direct proof of Theorem 1, based on the RAI theory.

Convergence issues. It should be noted that the convergence to $\mathcal{P}$ in (i) is asymptotic, meaning that the opinions of S-influenced agents do not actually terminate. Moreover, as the simulations above demonstrate (Fig. 1), this convergence can be quite slow, particularly in large networks.

Furthermore, this convergence does not guarantee that each individual opinion converges to a finite limit (some sufficient conditions for this will be considered in the next section). Note that, in the SCOD model with a general set $\mathcal{O}$, opinions can, in general, exhibit periodic oscillations in the presence of several stubborn agents[8]. However, we have not observed such trajectories in our experiments with the MHK-S, where the confidence sets are defined by norms. The convergence of S-influenced agents' opinions in an arbitrary norm thus remains a non-trivial open problem.

Analyzing the proof in [19], one observes that the convergence of S-influenced agents' opinions is equivalent to the fact that these opinions are "asymptotically stationary" in the sense that their successive increments vanish as time progresses: $\|\xi^i(t+1) - \xi^i(t)\| \to 0$ as $t \to \infty$. This non-trivial property has been established for the case of MHK-S based on the *Euclidean norm* [14], [19], where, in fact, the "kinetic energy" of the opinions proves to be finite:

$$\sum\nolimits_{t=0}^{\infty} \|\xi^i(t+1) - \xi^i(t)\|^2 < \infty.$$

[7]The connections between opinion dynamics and containment control have been well studied in the context of the Friedkin–Johnsen and Taylor models on static graphs [22]. In bounded confidence models, containment control strategies have been proposed to steer the group's opinions into a predefined range [23]; however, these models involve scalar opinions, and the problem is considerably different from the one addressed here.

[8]This shows, in particular, that general estimates of the "s-energy" [18] in dynamics of iterative averaging ("influence systems") are thus not easily extendable to scenarios involving multiple stubborn agents.

To the best of our knowledge, the validity of this fact for the MHK-S model based on a non-Euclidean norm (even the ℓ_p-norm with $p \neq 2$) remains an open problem, as does the convergence of S-influenced agents' opinions to stationary values. However, it is possible to prove such convergence for certain special allocations of the stubborn opinions. A sufficient criterion of this kind is provided in the next section.

IV. On Convergence of S-Influenced Agents

One situation in which Theorem 1 entails the convergence of all S-influenced agents' opinions is the case where $\mathcal{P}$ is a singleton, that is, all stubborn agents share the same opinion. This case has been studied (for a more general SCOD model) in [11]; an alternative (yet much more complicated proof) can be given by using the method of s-energy (similar to the analysis of the HK models with truth seekers in [18]).

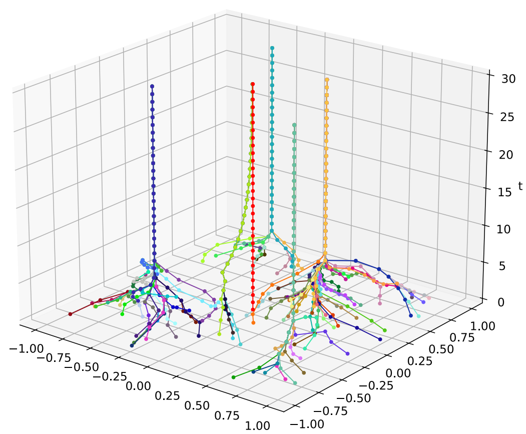

Fig. 3. The Unique Stubborn Agent Attracts S-Influenced Agents

The convergence of opinions in the case of one stubborn agent is illustrated in Fig. 3. In this experiment, $R = 0.5$ and the initial opinions of 99 regular agents are uniformly sampled from $[-1, 1]^2$. After 5 steps, these agents form four clusters of S-independent agents and one cluster of S-influenced agents (comprising 5 members), which converges to the stubborn agent's opinion (at the origin, shown in red).

A natural extension is to consider the case where the set $\mathcal{P}$ has a small diameter as shown by the following.

Corollary 1: Suppose that $\mathcal{V}_s \neq \emptyset$ and the opinions of stubborn agents are sufficiently close to each other, so that[9]

$$\|\xi^k - \xi^m\| < R \quad \forall\, k, m \in \mathcal{V}_s.$$

Then, the opinions of all S-influenced agents converge to the barycenter of stubborn agents' opinions

$$\bar{\xi}^c = \frac{1}{|\mathcal{V}_s|} \sum\nolimits_{k \in \mathcal{V}_s} \xi^k \in \mathbb{R}^d.$$

Proof: Since the opinions of S-independent agents no longer affect those of S-influenced agents after a transient period, we may, without loss of generality, assume that all regular agents are S-influenced. Statement (i) of Theorem 1 now implies that, for sufficiently large t, $\mathcal{N}_i(t) = \mathcal{V}$. Indeed, since all opinions $\xi^i(t), i \in \mathcal{V}_r$ converge to the set $\mathcal{P}$, whose diameter is less than R, it follows that for sufficiently large t, every pair of regular agents is confident in each other's opinions. For the same reason, each regular agent is confident in the stubborn agents' opinions. Therefore, for large t, the regular agents merge into a single cluster, and their common opinion $\xi(t) \triangleq \xi^i(t)$ obeys the linear equation

$$\xi(t+1) = \frac{|\mathcal{V}_r|}{|\mathcal{V}|}\xi(t) + \frac{1}{|\mathcal{V}|}\sum\nolimits_{k \in \mathcal{V}_s} \xi^k = \frac{|\mathcal{V}_r|\xi(t) + |\mathcal{V}_s|\bar{\xi}^c}{|\mathcal{V}|}.$$

The statement of Corollary now follows straightforwardly. ■

[9]Here the norm is supposed to be same as in the definition of the MHK-S.

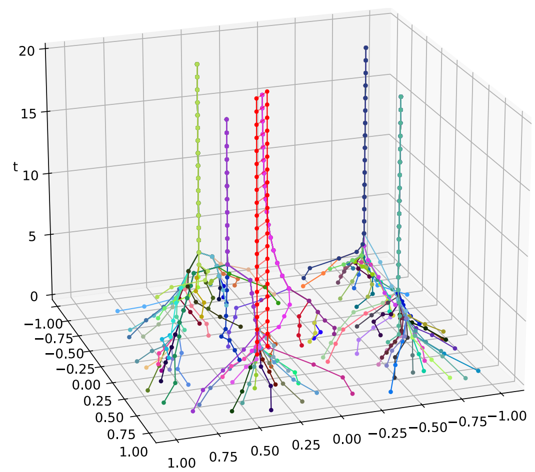

Fig. 4. Convergence to the Barycenter of Stubborn Agents' Opinions

Corollary 1 is illustrated in Fig. 4. In this experiment, two stubborn agents have coordinates $(0.1, 0.1)$ and $(0.2, 0.2)$, whereas $R = 0.5$. The small cluster of 6 S-influenced agents converges to the mean value of the stubborn agents' opinions.

We now consider a more complex scenario in which the stubborn agents are partitioned into several groups, each satisfying the assumptions of Corollary 1, and where the distance between any two groups is large enough. To formalize this situation, we introduce the following assumption.

Assumption 1: The set of stubborn agents $\mathcal{V}_s$ consists of several disjoint groups $\mathcal{V}_s = \mathcal{V}_s^1 \cup \mathcal{V}_s^2 \ldots \cup \mathcal{V}_s^p$, such that

(a) opinions in each group $\mathcal{V}_s^\ell$ are "close" in the sense that

$$\|\xi^i - \xi^j\| < R \quad \forall\, i, j \in \mathcal{V}_s^\ell \quad \forall \ell = 1, \ldots, p.$$

(b) at the same time, the convex hulls spanned by each group's opinions $\mathcal{P}^\ell = \mathrm{conv}\{\xi^i\}_{i \in \mathcal{V}_s^\ell}$ are separated by a distance of at least $(n_r + 1)R$, where $n_r = |\mathcal{V}_r|$.

Theorem 2: Under Assumption 1, the opinion of every S-influenced agent converges to one of the barycenters

$$\bar{\xi}^{c,\ell} = \frac{1}{|\mathcal{V}_s^\ell|} \sum\nolimits_{k \in \mathcal{V}_s^\ell} \xi^k \in \mathbb{R}^d.$$

Proof: Considering a solution $\Xi(t)$, without loss of generality, we again assume that all regular agents are S-influenced. For each group of stubborn agents, $\ell = 1, 2, \ldots, p$, we construct a sequence of sets $I^\ell(t)$ satisfying

$$\mathcal{V}_s^\ell \subseteq I^\ell(t) \subset I^\ell(t+1) \subseteq \mathcal{V},$$

with the following properties. First, for any $\ell \neq m$, the sets are disjoint, i.e., $I^\ell(t) \cap I^m(t) = \emptyset$. Second, the opinions of regular agents in $I^\ell(t)$ are "close enough" to the polytope $\mathcal{P}^\ell$; formally, for all $t \geq 0$ and every $i \in I^\ell(t)$, the inequality

$$\mathrm{dist}(\xi^i(t), \mathcal{P}^\ell) \leq R\,|I^\ell(t) \cap \mathcal{V}_r|$$

holds. We formally define $I^\ell(0) = \mathcal{V}_s^\ell$ for all $\ell = 1, \dots, p$.

At each time t, we examine the graph $\mathcal{G}(\Xi(t))$ and identify all nodes i that are connected to $I^\ell(t)$. The collection of these nodes forms the set $I^\ell(t+1) \supseteq I^\ell(t)$ (remember that each node is connected to itself). Notice that if a new agent $j \in I^\ell(t+1) \setminus I^\ell(t)$ is added, then by construction we have

$$\mathrm{dist}(\xi^j(t), \mathcal{P}^\ell) \leq R + \mathrm{dist}(\xi^i(t), \mathcal{P}^\ell) \leq R\,|I^\ell(t+1) \cap \mathcal{V}_r|,$$

where $i \in I^\ell(t)$ is one of the nodes to which j is connected at time t. Furthermore, such an agent j cannot be included in two different sets $I^\ell(t+1)$ and $I^m(t+1)$ simultaneously. Otherwise, one would have

$$\mathrm{dist}(\mathcal{P}^\ell, \mathcal{P}^m) \leq 2R + R\left|(I^\ell(t) \cup I^m(t)) \cap \mathcal{V}_r\right| \leq (2+n_r-1)R,$$

which contradicts Assumption 1. Hence, the sets $I^\ell(t+1)$ for different ℓ remain disjoint. Moreover, since at time t agents in $I^\ell(t)$ are influenced only by those in $I^\ell(t+1)$, equation (3) implies that the second condition holds:

$$\begin{aligned}\mathrm{dist}(\xi^i(t+1), \mathcal{P}^\ell) &\leq \max\nolimits_{j \in I^\ell(t+1)} \mathrm{dist}(\xi^j(t), \mathcal{P}^\ell) \\ &\leq R\,|I^\ell(t+1) \cap \mathcal{V}_r|.\end{aligned}$$

Clearly, after some step t_* the sets I^ℓ cease to expand, i.e., $I^\ell(t+1) = I^\ell(t)$ for all $t \geq t_*$. By construction, each subgroup of regular agents, defined as $\mathcal{V}_r^\ell \triangleq I^\ell(t_*) \cap \mathcal{V}_r$, is not influenced by any agent from $\mathcal{V} \setminus I^\ell(t_*)$—in particular, not by stubborn agents outside $\mathcal{V}_s^\ell$. Hence, the opinions of agents in $I^\ell(t_*) = \mathcal{V}_r^\ell \cup \mathcal{V}_s^\ell$ form a smaller MHK-S model that satisfies the conditions of Corollary 1. ■

Conclusion

We have extended the analysis of bounded confidence opinion dynamics to a multidimensional Hegselmann–Krause model with stubborn agents (MHK-S), where confidence sets are defined by general, non-Euclidean norms. We establish a behavioral dichotomy: the opinions of agents not persistently influenced by stubborn individuals terminate in finite time, whereas those under persistent influence converge asymptotically to the convex hull of the stubborn agents' opinions. Moreover, when the stubborn opinions are closely aligned, the influenced agents converge to their barycenter. These results further extend to scenarios involving multiple, well-separated clusters of stubborn agents.

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