Equivalence in the sense of time optimality for nonlinear systems with output

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***Abstract*— We describe a generalization of the connection of homogeneous approximation and approximation in the sense of time optimality for nonlinear control systems to the class of nonlinear control systems with output of arbitrary dimension. As the main tool, we use the representation of a nonlinear system linear in control as a series of iterated integrals and apply the free algebraic approach, which allows us to consider the problem of homogeneous approximation in algebraic and combinatorial terms.**

Keywords: nonlinear systems; homogeneous approximation; time-optimal control problem.

I. Introduction and background

A. Control systems and series of iterated integrals

Let us consider nonlinear control systems with output

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \quad y = h(x), \tag{1}$$

where $X_1(x), \ldots, X_m(x)$ are real analytic vector fields in a neighborhood of the origin $U(0) \subset \mathbb{R}^n$ and $h(x)$ is a real analytic nonzero map $h : U(0) \to \mathbb{R}^p$. Below we consider trajectories starting at the origin, $x(0) = 0$. As is well known [1], the output can be expressed in a series form

$$y(\theta) = c_\varnothing + \sum_{k=1}^{\infty} \sum_{1 \le i_1, \ldots, i_k \le m} c_{i_1 \ldots i_k} \eta_{i_1 \ldots i_k}(\theta, u), \tag{2}$$

where

$$\eta_{i_1 \ldots i_k}(\theta, u) = \\ = \int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_2 d\tau_1 \tag{3}$$

are iterated integrals and $c_{i_1 \ldots i_k} \in \mathbb{R}^p$ are constant vectors that can be found as

$$c_{i_1 \ldots i_k} = X_{i_k} \cdots X_{i_1} h(0). \tag{4}$$

Here X_i stands for a differential operator of the first order, $X_i \psi(x) = \psi'(x) X_i(x)$. Below we admit controls from the unit ball B^θ of the space $L_\infty([0, \theta]; \mathbb{R}^m)$; namely, we assume that $\sum_{i=1}^{m} u_i^2(t) \le 1$ a. e.

Iterated integrals are linearly independent functionals on the set B^θ, hence, the linear span of them (over $\mathbb{R}$) form a free associative algebra with the formal algebraic operation taking a pair $(\eta_{i_1 \ldots i_k}(\theta, u), \eta_{j_1 \ldots j_q}(\theta, u))$ to $\eta_{i_1 \ldots i_k j_1 \ldots j_q}(\theta, u)$. Then the series (2) can be thought of as a linear map from this algebra to $\mathbb{R}^p$.

B. Series in abstract free associative algebra

However, it is more convenient to introduce an abstract free associative algebra generated by m independent elements $\eta_1, \ldots, \eta_m$,

$$\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}^k,$$

where

$$\mathcal{F}^k = \mathrm{Lin}\{\eta_{i_1 \ldots i_k} : \ 1 \le i_1, \ldots, i_k \le m\}, \ k \ge 1,$$

with the algebraic operation

$$\eta_{i_1 \ldots i_k} \eta_{j_1 \ldots j_q} = \eta_{i_1 \ldots i_k j_1 \ldots j_q},$$

and, instead of the series (2), consider its abstract analogue,

$$S = c_\varnothing + \sum_{k=1}^{\infty} \sum_{1 \le i_1, \ldots, i_k \le m} c_{i_1 \ldots i_k} \eta_{i_1 \ldots i_k}. \tag{5}$$

Sometimes it is more convenient to consider a unitary algebra $\mathcal{F}^e = \mathcal{F} + \mathbb{R}$, then the constant term $c_\varnothing$ can be thought of as a coefficient of the unit. We also introduce the *inner product* in $\mathcal{F}$ assuming that $\eta_{i_1 \ldots i_k}$ form an orthonormal basis.

Notice that the series S generates the (linear) map $c : \mathcal{F}^e \to \mathbb{R}^p$ defined on basis elements as $c(\eta_{i_1 \ldots i_k}) = c_{i_1 \ldots i_k}$ and $c(1) = c_\varnothing$.

On the other hand, let us consider an arbitrary series of elements $\eta_{i_1 \ldots i_k}$ with vector coefficients. It *can* correspond to a system of the form (1) (that is, its coefficients can have the form (4) for some X_i and h) under some additional conditions called *realizability conditions* [1]. In order to formulate them, let us consider the free *Lie algebra* generated by elements $\eta_1, \ldots, \eta_m$,

$$\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^k,$$

where

$$\mathcal{L}^1 = \mathrm{Lin}\{\eta_1, \ldots, \eta_m\},$$
$$\mathcal{L}^k = \mathrm{Lin}\{[\ell, a] : \ell \in \mathcal{L}^1, a \in \mathcal{L}^{k-1}\}, \ k \ge 2,$$

and the Lie bracket is defined as a commutator, $[\ell_1, \ell_2] = \ell_1 \ell_2 - \ell_2 \ell_1$. Let us consider the map F_c defined on $\mathcal{L}$ that takes an element $\ell \in \mathcal{L}$ to the series

$$F_c(\ell) = c(\ell) + \sum_{k=1}^{\infty} \sum_{1 \le i_1, \ldots, i_k \le m} c(\eta_{i_1 \ldots i_k} \ell) \eta_{i_1 \ldots i_k}.$$

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The *Lie rank* [1] of the series S is defined as

$$\rho_L(S) = \dim \{F_c(\ell) : \ell \in \mathcal{L}\} .$$

Then, the realizability conditions are as follows: there exist $C, C_1 > 0$ such that $\|c_{i_1\ldots i_k}\| \le C_1 k! C^k$ for any $k \ge 1$ and $\rho_L(S) < \infty$. Moreover, $n = \rho_L(S)$ is the minimal dimension of a *realizing system*; the realizing system of dimension n (the *minimal realization*) is unique up to a change of variables [1].

If a real analytic change of variables is applied to the system and/or we want to consider a series for a real analytic function of the output, then we can manipulate with the original series finding the result of maps in algebraic terms. To this end, the *shuffle product* operation in $\mathcal{F}^e$ can be used, which is defined as

$$\eta_{i_1\ldots i_k} \sqcup\!\sqcup\, \eta_{j_1\ldots j_p} = \eta_{i_1}(\eta_{i_2\ldots i_k} \sqcup\!\sqcup\, \eta_{j_1\ldots j_p}) + \eta_{j_1}(\eta_{i_1\ldots i_k} \sqcup\!\sqcup\, \eta_{j_2\ldots j_p})$$

where $1 \sqcup\!\sqcup\, a = a \sqcup\!\sqcup\, 1 = a$ for any $a \in \mathcal{F}^e$. This operation corresponds to multiplication of iterated integrals as functionals. Namely, if we find the shuffle product $\eta_{i_1\ldots i_k} \sqcup\!\sqcup\, \eta_{j_1\ldots j_p}$ and consider the corresponding linear combination of iterated integrals, then we get the same functional as we get when we rewrite the product of two iterated integrals

$$\eta_{i_1\ldots i_k}(\theta, u) \cdot \eta_{j_1\ldots j_p}(\theta, u)$$

as a linear combination of iterated integrals, i.e., integrals over simplexes of the form (3).

As a result, instead of finding (real analytic) functions of iterated integrals, we can find corresponding functions in $\mathcal{F}$ using shuffle product instead of the product of integrals. Thus, many questions related to systems of the form (1) can be answered by use of algebraic and combinatorial technique [2]. In particular, this allows applying computational tools to find homogeneous approximations [3].

C. Homogeneous approximation in the case of trivial output

In the particular case, when the output is trivial, $h(x) = x$, and the series satisfies the Rashevsky-Chow condition,

$$\dim c(\mathcal{L}) = n, \tag{6}$$

the concept of homogeneous approximation is well known and deeply studied [4] and is widely used [5]. In algebraic terms, it can be introduced as follows [6].

Let us consider a series of the form (5) with $c_\varnothing = 0$ and $c_{i_1\ldots i_k} \in \mathbb{R}^n$ satisfying the realizability conditions and the Rashevsky-Chow condition (6). There exist integers $w_1 \le \cdots \le w_n$ and a (polynomial) change of variables $y = F(x)$ such that the series $F(S)$ takes the form

$$F(S) = (a_1 + R_1, \ldots, a_n + R_n)^\top ,$$

where $a_j \in \mathcal{F}^{w_j}$, $R_j \in \sum_{k \ge w_j+1} \mathcal{F}^k$, and the series

$$\widehat{S} = (a_1, \ldots, a_n)^\top$$

also satisfies the realizability and Rashevsky-Chow conditions. This means that there exists a realizing system for $\widehat{S}$, that is, the system

$$\dot{x} = \sum_{i=1}^m \widehat{X}_i(x) u_i,$$

such that its series equals $\widehat{S}$. Therefore, componentwise $x_j(\theta) = a_j(\theta)$, $j = 1, \ldots, n$, where in the left hand side $x_j(\theta)$ means the end of the trajectory of the system (1) at (any) time moment θ whereas in the right hand side $a_j(\theta)$ equals the linear combination of iterated integrals of the form (3) corresponding to elements from the abstract algebra $\mathcal{F}$ included in a_j.

This system, which is defined uniquely up to polynomial changes of variables, is called *a homogeneous approximation* for the original system (1) [6]. This algebraic way leads to the same concept of a homogeneous approximation as was obtained by other approach [4].

The elements a_i can be explicitly obtained algebraically. To this end, we introduce the so-called *core Lie subalgebra*

$$\mathcal{L}_S = \sum_{k=1}^\infty \mathcal{P}^k, \tag{7}$$

where $\mathcal{P}^1 = \mathrm{Lin}\{\ell \in \mathcal{L}^1 : c(\ell) = 0\}$,

$$\mathcal{P}^k = \mathrm{Lin}\{\ell \in \mathcal{L}^k : c(\ell) \in c(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})\},\ k \ge 2.$$

In the considered case, $\mathcal{L}_S$ is of codimension n (in $\mathcal{L}$). It turns out that two systems have the same homogeneous approximation if and only if they have the same core Lie subalgebras.

In order to find the elements a_i, one can proceed as follows. Consider the *left ideal* generated by $\mathcal{L}_S$,

$$\mathcal{J}_S = \mathrm{Lin}\{a\ell : a \in \mathcal{F}^e, \ell \in \mathcal{L}_S\}.$$

Consider any set of linearly independent elements $\ell_i \in \mathcal{L}^{w_i}$, $i = 1, \ldots, n$, such that $w_i \le w_j$ if $i < j$ and $\mathcal{L} = \mathcal{L}_S + \mathrm{Lin}\{\ell_1, \ldots, \ell_n\}$. Then the elements a_i can be chosen as $a_i = \widetilde{\ell}_i$, where $\widetilde{\ell}_i$ are the orthogonal projections of ℓ_i on the subspace $\mathcal{J}_S^\perp$ (orthogonal complement of $\mathcal{J}_S$) [6]. Elements a_i are defined up to shuffle polynomials of $\widetilde{\ell}_j$. Another way to describe a possible form of a_i is related to the Poincaré-Birkhoff-Witt basis and the dual basis in $\mathcal{F}$ [6].

The key question is in what sense the homogeneous approximation approximates the original system (1) [4]. As was shown in [6], under some conditions the homogeneous approximation approximates the original system *in the sense of time optimality*. More specifically, let us consider the time-optimal problem of minimization θ subject to

$$\dot{x} = \sum_{i=1}^m X_i(x) u_i,\ x(0) = 0,\ x(\theta) = s, \sum_{i=1}^m u_i^2(t) \le 1 \text{ a.e.},$$

where the end point s is given, and the same problem for the homogeneous approximation. Then, after some change of variables in the original system, the optimal times and optimal controls for the original system and for its homogeneous approximation are in a certain sense equivalent in a neighborhood of the origin; details can be found in [6].

D. Systems with nontrivial output

Now, let us consider the general case, when the output $h(x)$ is arbitrary (but non-constant). Without loss of generality we assume that $h(0) = 0$. Algebraically, this means that we consider series of the form (5) with coefficients from $\mathbb{R}^p$ (and $c_\varnothing = 0$) satisfying the realizability conditions. Let $\rho_L(S) = n$. Then there exists an n-dimensional realizing system (i.e., minimal realization), and one can consider its homogeneous approximation.

In [7] we showed that the core Lie subalgebra of the minimal realization and, consequently, its homogeneous approximation can be found without finding the realizing system itself, by use of algebraic constructions only. Namely, the core Lie subalgebra of the minimal realization equals (7), where the subspaces $\mathcal{P}^k$ are defined as

$$\mathcal{P}^1 = \{\ell \in \mathcal{L}^1 : c(a\ell) = 0 \text{ for any } a \in \mathcal{F}^e\},$$

$$\mathcal{P}^k = \{\ell \in \mathcal{L}^k : \text{there exists } \ell' \in \mathcal{L}^1 + \cdots + \mathcal{L}^{k-1} \text{ such that } c(a(\ell - \ell')) = 0 \text{ for any } a \in \mathcal{F}^e\},\ k \geq 2.$$

However, much more interesting question is about a *homogeneous approximation of the series S itself.*

E. Systems with one-dimensional output

The case when the output is one-dimensional, $p = 1$, was considered in [7]. Let us consider a (nonzero) series (5) with one-dimensional coefficients such that $c_\varnothing = 0$ and suppose that the realizability conditions are satisfied and $\rho_L(S) = n$. Denote $r = \min\{k : c(\mathcal{F}^k) \neq 0\}$. Then we say that

$$\widehat{S} = \sum_{1 \leq i_1, \ldots, i_r \leq m} c_{i_1 \ldots i_r} \eta_{i_1 \ldots i_r}$$

is *a homogeneous approximation for the series S*. Actually, $\widehat{S}$ is a finite linear combination of elements of the same "length", that is, $\widehat{S}$ is homogeneous.

Obviously, $\widehat{S}$ satisfies the realizability conditions, that is, $\rho_L(\widehat{S}) < \infty$. One can show [7] that $\rho_L(\widehat{S}) \leq \rho_L(S) = n$, hence, the minimal realization of $\widehat{S}$ is of dimension no greater than n. Moreover, the following relation between the core Lie subalgebras holds, $\mathcal{L}_S \subset \mathcal{L}_{\widehat{S}}$.

It turns out that the core Lie subalgebra $\mathcal{L}_{\widehat{S}}$ admits the following description. Let us consider the left ideal generated by $\mathcal{L}_{\widehat{S}}$, that is, $\mathcal{J}_{\widehat{S}} = \mathrm{Lin}\{a\ell : a \in \mathcal{F}^e, \ell \in \mathcal{L}_{\widehat{S}}\}$. We say that a left ideal $\mathcal{J}'$ is *Lie generated* if there exists a graded Lie subalgebra of finite codimension $\mathcal{L}' \subset \mathcal{L}$ such that $\mathcal{J}' = \mathrm{Lin}\{a\ell : a \in \mathcal{F}^e, \ell \in \mathcal{L}'\}$. Then $\mathcal{J}_{\widehat{S}}$ coinsides with the *maximal Lie generated left ideal orthogonal to $\widehat{S}$* [7].

In [7], we studied approximation in the sense of time optimality for the case of one-dimensional series. We recall the main result.

Let S be a series with one-dimensional coefficients satisfying the realizability conditions and $\rho_L(S) = n$. We consider the following time-optimal problem for its minimal realization, that is, to minimize θ subject to

$$\dot{x} = \sum_{i=1}^m X_i(x)u_i, x(0) = 0, h(x(\theta)) = s, \sum_{i=1}^m u_i^2(t) \leq 1 \text{ a.e.},$$

where $s \neq 0$ is fixed. We denote the optimal time by θ_s^* and the set of optimal controls by U_s^* (if it is nonempty).

Let us consider the homogeneous approximation $\widehat{S}$ of the series S and the time-optimal problem for its realization: minimize θ subject to

$$\dot{x} = \sum_{i=1}^m \widehat{X}_i(x)u_i, x(0) = 0, \widehat{h}(x(\theta)) = s, \sum_{i=1}^m u_i^2(t) \leq 1 \text{ a.e.}$$

We denote the optimal time by $\widehat{\theta}_s^*$ and the set of optimal controls by $\widehat{U}_s^*$ (if it is nonempty). Without loss of generality we assume that the series for the system and its homogeneous approximation are such that $S_j = a_j + R_j$, $\widehat{S}_j = a_j$, where $a_j \in \mathcal{F}^{w_j}$ and $R_j \in \sum_{k \geq w_j + 1} \mathcal{F}^k$.

In [7], we obtained the following result.

(i) Suppose the latter problem has a solution for $s = 1$. Then there exists $\varepsilon > 0$ such that for any $s \in (0, \varepsilon)$ the former problem also has a solution and

$$\frac{\theta_s^*}{\widehat{\theta}_s^*} \to 1 \text{ as } s \to +0.$$

(ii) Suppose, in addition, that the set of optimal controls $\widehat{U}_1^*$ is finite. Then for any sequence of scalars such that $s_{(k)} \to +0$ as $k \to +\infty$, and any sequence of optimal controls $u_{s_{(k)}}^*(t) \in U_{s_{(k)}}^*$ there exists a sequence $\widehat{u}_{s_{(k)}}^*(t) \in \widehat{U}_{s_{(k)}}^*$ such that for any $i = 1, \ldots, m$

$$\int_0^1 \left| u_{s_{(k)}i}^*(t\theta_{s_{(k)}}^*) - \widehat{u}_{s_{(k)}i}^*(t\widehat{\theta}_{s_{(k)}}^*) \right| dt \to 0 \text{ as } k \to +\infty.$$

These properties mean that the optimal times and optimal controls for the original system and for its homogeneous approximation are equivalent in a neighborhood of the origin. We emphasize that the proof in [7] essentially uses the fact that the output is one-dimensional.

Such an approximation property of optimal controls seems to be rather complicated. This is due to non-uniqueness of the optimal control for both problems, which is typical for the case when the output is nontrivial. Under the uniqueness requirement, we could obtain a simpler result close to [6].

II. The main result

In [8], we considered the general case of output of an arbitrary dimension p. We introduced an algebraic description of the homogeneous approximation generalizing the one-dimensional case and the case of the trivial output. First, let us consider the set of all formal functions of p arguments from $\mathcal{F}$, which are defined as formal power series with respect to the shuffle multiplication. For any *one-dimensional* series S, by $S_{\min}$ we denote the sum of terms of minimal length. That is, if $r = \min\{k : c(\mathcal{F}^k) \neq 0\}$, then

$$S_{\min} = \sum_{1 \leq i_1, \ldots, i_r \leq m} c_{i_1 \ldots i_r} \eta_{i_1 \ldots i_r}.$$

Let us introduce the set $N_S \subset \mathcal{F}$ of all elements of the form $(f(S))_{\min}$, where f is an arbitrary formal function. Finally, let us consider the set of all Lie generated left ideals that are orthogonal to the set N_S, and introduce the maximal ideal $\mathcal{J}_S^{\max}$ from this set. Since $\mathcal{J}_S$ also belongs to this set, we have $\mathcal{J}_S \subset \mathcal{J}_S^{\max}$. Let $\mathcal{J}_S^{\max}$ be generated by the Lie subalgebra $\mathcal{L}_S^{\max}$.

Generalizing the approach described in the previous section, we can show [8] that there exists a number $q \le p$, numbers $w_1 \le \cdots \le w_q$, an invertible mapping F, and elements $\widehat{a}_i \in \mathcal{F}^{w_i}$ such that

$$F(S) = (\widehat{a}_1 + R_1, \ldots, \widehat{a}_q + R_q, 0, \ldots, 0)^\top ,$$

where $\widehat{a}_j \in \mathcal{F}^{w_j}$, any of $\widehat{a}_j$ is not a shuffle polynomial of the others, and $R_j \in \sum_{k \ge w_j+1} \mathcal{F}^k$. This result allows us to call the series $\widehat{S} = (\widehat{a}_1, \ldots, \widehat{a}_q)^\top$ *a homogeneous approximation of* S. Clearly, the requirement that any of $\widehat{a}_j$ is not a shuffle polynomial of the others is essential here. We note that the elements $\widehat{a}_1, \ldots, \widehat{a}_q$ satisfy the following property: any element of N_S equals a shuffle polynomial of $\widehat{a}_1, \ldots, \widehat{a}_q$.

We say that two series are *algebraically equivalent* if they have the same homogeneous approximation. We obtained the algebraic description of this property [8]: two series S^1 and S^2 are algebraically equivalent if and only if $N_{S^1} = N_{S^2}$.

However, the question on approximation *in the sense of time optimality* remains open for the case of multi-dimensional output. The results from [8] allow us to apply the steps of the proof from [6] and generalize the approximation theorem from [7] mentioned above.

Let S be a series with p-dimensional coefficients satisfying the realizability conditions and $\rho_L(S) = n$. We consider the time-optimal problem for the minimal realization: minimize θ subject to

$$\dot{x} = \sum_{i=1}^m X_i(x)u_i, x(0) = 0, h(x(\theta)) = s, \sum_{i=1}^m u_i^2(t) \le 1 \text{ a. e.}, \tag{8}$$

where $s \ne 0$ is given. Here $s \in \mathbb{R}^p$ for $p > 1$. Denote the optimal time by θ_s^* and the set of optimal controls by U_s^* (if it is nonempty). Without loss of generality we suppose that the system is given in coordinates such that its series has the form

$$S = (\widehat{a}_1 + R_1, \ldots, \widehat{a}_q + R_q, 0, \ldots, 0)^\top ,$$

then $s = (s_1, \ldots, s_q, 0, \ldots, 0)^\top$. Denote $s' = (s_1, \ldots, s_q)^\top$ and consider the problem of minimizing θ subject to

$$\dot{x} = \sum_{i=1}^m \widehat{X}_i(x)u_i, x(0) = 0, \widehat{h}(x(\theta)) = s', \sum_{i=1}^m u_i^2(t) \le 1 \text{ a.e.}, \tag{9}$$

where the system is a homogeneous approximation of the system from (8), that is, its series equals

$$S = (\widehat{a}_1, \ldots, \widehat{a}_q)^\top .$$

Denote the optimal time by $\widehat{\theta}_{s'}^*$ and the set of optimal controls by $\widehat{U}_{s'}^*$ (if it is nonempty).

Then the results mentioned above suggest the following theorem.

Theorem 1: Suppose there exists a domain $\Omega \subset \mathbb{R}^q \backslash \{0\}$ such that $0 \in \overline{\Omega}$ and for any $s' \in \Omega$ there exists a solution of the problem (9).

(i) Then there exists a domain $\Omega' \subset \Omega$ such that $0 \in \overline{\Omega'}$ and for any $s' \in \Omega'$ the problem (8) also has a solution (for $s = (s'_1, \ldots, s'_q, 0, \ldots, 0)^\top$) and

$$\frac{\theta_s^*}{\widehat{\theta}_{s'}^*} \to 1 \text{ as } \ s' \to 0, \ s' \in \Omega'.$$

(ii) Suppose, in addition, that for any $s' \in \Omega$ the set of optimal controls $\widehat{U}_{s'}^*$ is finite. Then for any sequence $s'_{(k)} \in \Omega'$ such that $s'_{(k)} \to 0$ as $k \to +\infty$ and any sequence of optimal controls $u_{s_{(k)}}^*(t) \in U_{s_{(k)}}^*$ there exists a sequence $\widehat{u}_{s'_{(k)}}^*(t) \in \widehat{U}_{s'_{(k)}}^*$ such that for any $i = 1, \ldots, m$

$$\int_0^1 \left| u_{s_{(k)}i}^*(t\theta_{s_{(k)}}^*) - \widehat{u}_{s'_{(k)}i}^*(t\widehat{\theta}_{s'_{(k)}}^*) \right| dt \to 0 \text{ as } \ k \to +\infty.$$

Actually, the proof, which follows [6], allows us to describe domains Ω' more specifically.

Example. Let us consider the following series

$$S = (\eta_2 + \eta_{21}, \ \eta_{22} + \eta_{221})^\top .$$

One can check that $\rho_L(S) = 3$; a minimal realization can be chosen as

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2 + x_1 u_2, \quad \dot{x}_3 = x_2 u_2 \tag{10}$$

with two-dimensional output $h(x) = (x_2, x_3)^\top$. A homogeneous approximation of the series S can be chosen as

$$\widehat{S} = (\eta_2, \ \eta_{221} + \eta_{212})^\top ;$$

it corresponds to the *homogeneous* system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 x_2 u_2 \tag{11}$$

with the output $\widehat{h}(x) = (x_2, x_3)^\top$. So, the solution of the homogeneous time-optimal problem (9) for the system (11) approximates the solution of the time-optimal problem (8) for the system (10) in the sense of Theorem 1.

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