Solving the time-optimal control problem for nonlinear non-autonomous linearizable systems*

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***Abstract*— We present conditions under which the time-optimal control problem for a nonlinear non-autonomous linearizable system can be solved by the method of successive approximations, at each step of which a power Markov moment min-problem is solved. The proposed method can be efficiently implemented by use of symbolic and numerical calculations.**

Keywords: Nonlinear control system, Linearizablity problem, Linear control system with analytic matrices, Method of successive approximations, Power Markov moment min-problem with gaps.

I. INTRODUCTION

The linearizability problem is an important issue for nonlinear control theory. The first results were obtained in 1973 by A. Krener [1] and V.I. Korobov [2]; they had a great impact on subsequent research. Later, conditions of linearzability for systems from different classes were proposed. During long time, the main focus was on smooth systems (from the class C^∞) [3], [4], [5], [6]. Later, the ideas of V.I. Korobov's paper [2] were developed and linearizability conditions were obtained for systems of the class C^1 [7]. Recently, an approach to studying linearzability of non-autonomous systems of the class C^1 was proposed [8].

If a nonlinear system turns out to be linearizable, well-developed methods from the linear control theory can be applied. However, to this end, it is not enough to check that linearizability conditions are met. First, we need to find a linearizing change of variables. Second, efficient methods of solving control problems for linear non-autonomous systems should be applied.

In this paper, we propose a method for solving the time-optimal control problem for non-autonomous linearizable systems with a single input.

Let us consider a control system

$$\dot{x} = f(t, x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^1, \tag{1}$$

and suppose that $f \in C^1([0, \beta] \times Q \times \mathbb{R})$ where $\beta > 0$ and $Q \subset \mathbb{R}^n$ is a neighborhood of the origin. We say that the system (1) is *locally analytically linearizable at the origin* if there exists a local change of variables $z = F(t, x) \in C^2([0, \beta] \times Q)$ such that the system in the new variables is linear, i.e., takes the form

$$\dot{z} = A(t)z + b(t)u, \tag{2}$$

where components of $A(t)$ and $b(t)$ are real analytic in $[0, \beta]$. Here and below under "local change of variables" we mean a map $F(t, x)$ that takes the origin to itself and is locally invertible w.r.t. x, i.e.,

$$F(t, 0) \equiv 0, \quad \det F_x(t, x) \neq 0, \quad (t, x) \in [0, \beta] \times Q,$$

where the sub-index means the derivative in x, that is, $F_x(t, x) = \frac{\partial F(t,x)}{\partial x}$. Clearly, this is true (maybe in a smaller neighborhood) if $\det F_x(0, 0) \neq 0$. As one can see (we explain this issue below), if the system (1) is locally analytically linearizable at the origin, then it has the control-affine form

$$\dot{x} = a(t, x) + b(t, x)u, \quad a(t, 0) \equiv 0. \tag{3}$$

The linearizability property can be used for solving the local controllability problem for the system (3): for two given points $x(0) = x^0$, $x(T) = x^1$, find $z^0 = F(0, x^0)$ and $z^1 = F(T, x^1)$ and then find a control $u(t)$ which steers the linear system (2) from z^0 to z^1 in the time T; then this control steers the system (3) from x^0 to x^1. In this paper we propose a method for solving the time-optimal control problem under the constraint $|u(t)| \leq 1$.

II. PROBLEM STATEMENT AND OUTLINE OF THE PAPER

Let us consider the time-optimal control problem of the form

$$\begin{aligned} &\dot{x} = a(t, x) + b(t, x)u, \quad a(t, 0) \equiv 0, \\ &x(0) = x^0, \ x(\theta) = 0, \quad |u(t)| \leq 1, \quad \theta \to \min, \end{aligned} \tag{4}$$

assuming that x^0 belongs to a sufficiently small neighborhood of the origin.

Our goals are:

1) to formulate explicit conditions under which the system (3) is locally analytically linearizable at the origin and, moreover, the corresponding linear system (2) admits an efficient method of solving the time-optimal control problem;

2) to describe the method of solving the time-optimal control problem (4) for such a nonlinear system of the form (3) directly, without finding a linearizing change of variables.

First of all, an efficient method of solving the linear time-optimal control problem should be involved. In Subsection III-A we recall known results related to systems of the form (2), where $A(t)$ and $b(t)$ are real analytic in a

*This work was financially supported by Polish National Science Centre grant no. 2017/25/B/ST1/01892.

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neighborhood of zero. For starting points from a neighborhood of the origin, an optimal control equals ± 1 and has no more than $n-1$ switchings. The direct substitution of such a control leads to a system of n nonlinear equations with n unknowns (switching times and the optimal time). However, under some conditions, the optimal control can be found by the method of successive approximations, at each step of which *a power Markov moment min-problem with gaps* is solved. The power Markov moment problem was originated in [9], a deep discussion can be found in [10]. The statement of the Markov moment min-problem and its application to the time-optimal control problem was proposed in [11], [12]; in many cases it admits an explicit solution.

Then, in Subsection III-B, we recall some recent results on *linearizability conditions for non-autonomous systems* proposed in [8]. Additionally to linearizability conditions known since [1], [4] and generalized to systems of the class C^1 in [7], [13], in the non-autonomous case some new conditions arise, see [14] and [15] for further discussion.

Finally, in Section IV we combine the known results mentioned above and formulate the main result of the paper (Theorem 3), which gives the method for solving the time-optimal control problem for non-autonomous linearizable systems. This method can be effectively used for numerical application; we demonstrate it by an illustrative example in Section V.

Main results of this paper were presented as preprints [16], [17].

III. Background

A. Solving the time-optimal control problem for linear non-autonomous system

Consider a control system of the form

$$\dot{x} = A(t)x + b(t)u, \tag{5}$$

where the matrix $A(t)$ and the vector $b(t)$ are real analytic in some interval $[0,\beta]$. Let us suppose that a control $u(t)$ steers this system from the point x^0 to the origin in the time $\theta \in (0,\beta)$, i.e., $x(0)=x^0$, $x(\theta)=0$. Then

$$x^0 = -\int_0^\theta \Phi^{-1}(t)b(t)u(t)dt, \tag{6}$$

where the matrix $\Phi(t)$ is defined by $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(0)=I$. The vector function $\Phi^{-1}(t)b(t)$ is analytic; let us expand it into the Taylor series at $t=0$. One can easily prove by induction that the derivatives of $\Phi^{-1}(t)b(t)$ equal

$$(\Phi^{-1}(t)b(t))^{(i)} = \Phi^{-1}(t)\left(-A(t)+\tfrac{d}{dt}\right)^i b(t), \quad i \geq 0.$$

Hence, (6) implies

$$x^0 = -\sum_{i=0}^{\infty} L_i \int_0^\theta t^i u(t)dt, \tag{7}$$

where

$$L_i = \tfrac{1}{i!}\left(-A(t)+\tfrac{d}{dt}\right)^i b(t)|_{t=0}, \quad i \geq 0.$$

This means that the right hand side of (7) is a series of *power moments* of the function $u(t)$ with vector coefficients L_j. Suppose that the system (5) is controllable on $(0,\beta)$, then

$$\operatorname{rank}\{L_i\}_{i=0}^{\infty} = n. \tag{8}$$

Let $k_1,\ldots,k_n$ be the indices of the first n linearly independent vectors from the sequence $\{L_i\}_{i=0}^{\infty}$. Denote by L the matrix

$$L = (-L_{k_1},\ldots,-L_{k_n}).$$

Multiplying both sides of the equality (7) by L^{-1}, we get

$$(L^{-1}x^0)_i = \int_0^\theta t^{k_i}u(t)dt + \sum_{j=k_i+1}^{\infty} \alpha_{ji}\int_0^\theta t^j u(t)dt \tag{9}$$

for $i=1,\ldots,n$, where α_{ji} are components of the vector $-L^{-1}L_j$. Below we suppose $|u(t)| \leq 1$, then $|\int_0^\theta t^j u(t)dt| \leq \frac{1}{j+1}\theta^{j+1}$. This means that locally, for small θ, the first term in the right hand side of (9) is a "leading" one. Having this in mind, we consider the *power Markov moment min-problem with gaps* [11], [12], [18]

$$y_i = \int_0^\theta t^{k_i}u(t)dt, \quad i=1,\ldots,n, \quad |u(t)| \leq 1, \quad \theta \to \min. \tag{10}$$

As was shown in [19], the solution $(\widehat{\theta}(x^0), \widehat{u}(t;x^0))$ of the time-optimal control problem

$$\begin{aligned} &\dot{x} = A(t)x + b(t)u, \\ &x(0)=x^0,\ x(\theta)=0, \quad |u(t)| \leq 1, \quad \theta \to \min \end{aligned} \tag{11}$$

and the solution $(\theta(y), u(t;y))$ of the power Markov moment min-problem (10) for $y = L^{-1}x^0$ are *equivalent at the origin*, i.e.,

$$\frac{\widehat{\theta}(x^0)}{\theta(L^{-1}x^0)} \to 1, \quad \frac{1}{\theta}\int_0^\theta |\widehat{u}(t;x^0) - u(t;L^{-1}x^0)|dt \to 0$$

as $x^0 \to 0$, where $\theta = \min\{\widehat{\theta}(x^0), \theta(L^{-1}x^0)\}$.

Under some additional conditions this result can be strengthened, namely, a fixed-point iteration can be used for finding the solution [12]. In [19], the following theorem was proved.

Theorem 1: Consider the system (5) where $A(t)$ and $b(t)$ are real analytic in a neighborhood of zero and assume the condition (8) holds. Suppose also that

$$L_i = 0 \ \text{ for all } \ i < k_n \text{ such that } i \neq k_j,\ j=1,\ldots,n-1. \tag{12}$$

Then there exists a neighborhood $U(0)$ of the origin such that, for any $x^0 \in U(0)$ the solution $(\widehat{\theta}(x^0), \widehat{u}(t;x^0))$ of the time-optimal control problem (11) can be found as

$$\widehat{\theta}(x^0) = \lim_{r\to\infty}\theta(y^r), \quad \widehat{u}(t;x^0) = \lim_{r\to\infty} u(t;y^r), \tag{13}$$

where $(\theta(y), u(t;y))$ denotes the solution of the Markov moment min-problem (10) and the sequence $\{y^r\}_{r=0}^{\infty}$ is defined recursively as

$$y^0 = L^{-1}x^0$$

and

$$y^{r+1} = L^{-1}\left(x^0 + \int_0^{\theta(y^r)} \Phi^{-1}(t)b(t)u(t;y^r)dt\right) + y^r$$

for $r \geq 0$.

This result follows from the fact that under the condition (12) the map

$$\begin{aligned} y \mapsto \quad & L^{-1}\left(x^0 + \sum_{j\neq k_i} L_j \int_0^{\theta(y)} t^j u(t;y)dt\right) = \\ & = L^{-1}\left(x^0 + \int_0^{\theta(y)} \Phi^{-1}(t)b(t)u(t;y)dt\right) + y \end{aligned}$$

is a contraction in a neighborhood of the origin; if $\bar{y}$ is its fixed point, then the control $u(t;\bar{y})$ steers the system (5) to the origin in the time $\theta(\bar{y})$.

We notice that the control $u(t;\bar{y})$ solving the problem (10) equals ± 1 and has its switchings at the roots of some polynomial $p(t) = \sum_{i=1}^n \alpha_i t^{k_i}$. However, the functions $t^{k_1}, \ldots, t^{k_n}$ form a Chebyshev system on any interval $(0,\tau)$ [10]. This means that the polynomial $p(t)$ has no more than $n-1$ roots and therefore the control $u(t;\bar{y})$ equals ± 1 and has no more than $n-1$ switchings on the interval $(0,\theta(\bar{y}))$.

On the other hand, since the components of the vector function $\Phi^{-1}(t)b(t)$ are linearly independent analytic functions, they also form a Chebyshev system on any interval $(0,\tau)$ if τ is small enough [10], [19]. This implies that the control $u(t;\bar{y})$, which steers the system (5) to the origin and has no more than $n-1$ switchings, is time-optimal for the problem (11), see [12] for details.

In particular, if $k_i = i-1$, $i = 1,\ldots,n$, then the condition (12) is satisfied automatically. Moreover, in this case the moment problem (10) has no gaps, hence, it can be effectively and completely solved by the method described in [11]; see also [20] for additional comments and examples.

For the power Markov moment min-problem with gaps (10) of the general form, a deep study was conducted in [18]. One particular case of even gaps was treated in [21].

B. *Conditions of linearizability for non-autonomous systems*

In [8], linearizability conditions for nonlinear non-autonomous control systems were given; further analysis can be found in [14], [15]. In this subsection we formulate a direct corollary of these results related to a local statement of the problem.

Suppose that a system of the form (1) is locally linearizable, then for some change of variables $z = F(t,x)$ we have $\dot{z} = A(t)z + b(t)u$, that is, $F_t(t,x) + F_x(t,x)f(t,x,u) = A(t)F(t,x) + b(t)u$. Differentiating both sides by u, we get $F_x(t,x)f_u(t,x,u) = b(t)$, which gives $f_u(t,x,u) = (F_x(t,x))^{-1}b(t)$. This means that the derivative of f with respect to u does not depend on u, which implies $f(t,x,u) = a(t,x) + b(t,x)u$. Additionally, we require that the origin is an equilibrium of the system, i.e., $f(t,0,0) \equiv 0$, which implies $a(t,0) \equiv 0$. Hence, we deal with a control-affine system of the form (3), where $a(t,x)$ and $b(t,x)$ are defined in a neighborhood of the origin. The smoothness requirements for $a(t,x)$ and $b(t,x)$ are specified below.

Denote by $\mathcal{R}$ the following operator that acts on a vector function $c(t,x)$ by the rule

$$\mathcal{R}c(t,x) = c_t(t,x) + [a(t,x), c(t,x)],$$

where $[\cdot,\cdot]$ denotes the Lie bracket, $[a(t,x),c(t,x)] = c_x(t,x)a(t,x) - a_x(t,x)c(t,x)$, and the sub-indices t and x denote the derivatives w.r.t. t and x respectively. Introduce the following matrix

$$R(t,x) = (b(t,x), \mathcal{R}b(t,x), \ldots, \mathcal{R}^{n-1}b(t,x)).$$

Also we use the notation $k^{\underline{j}}$ for the *falling factorial*,

$$k^{\underline{j}} = k(k-1)\cdots(k-j+1),\ j \geq 1, \quad k^{\underline{0}} = 1.$$

Theorem 2: Consider a non-autonomous control system of the form (3), where $a(t,x) \in C^2([0,\beta]\times Q)$, $b(t,x) \in C^1([0,\beta]\times Q)$ for some $\beta > 0$ and some neighborhood of the origin, $0 \in Q \subset \mathbb{R}^n$. Suppose that all vector functions $\mathcal{R}^i b(t,x)$ for $1 \leq i \leq n$ exist and belong to the class $C^1([0,\beta]\times Q)$ and the following conditions are satisfied,

1) $[\mathcal{R}^i b(t,x), \mathcal{R}^j b(t,x)] = 0$ for $0 \leq i < j \leq n-1$, $(t,x) \in [0,\beta]\times Q$;
2) $\operatorname{rank} R(t,x) = n$ for $t \in (0,\beta]$ and $x \in Q$;
3) the vector function $R^{-1}(t,x)\mathcal{R}^n b(t,x)$ depends only on t, i.e.,
$$R^{-1}(t,x)\mathcal{R}^n b(t,x) = \gamma(t);$$
4) components of $\gamma(t)$ are analytic or meromorphic functions in $(0,\beta]$ with a pole at $t=0$ such that
$$\gamma_i(t) = \sum_{j=-n+i-1}^{\infty} \gamma_{i,j} t^j, \quad i = 1,\ldots,n,$$
the indicial equation
$$k^{\underline{n}} - \sum_{s=1}^n k^{\underline{n-s}} \gamma_{n-s+1,-s} = 0$$
has n integer nonnegative roots $0 \leq k_1 < \cdots < k_n$ and the rank of the matrix
$$\begin{pmatrix} V_{k_1+1,k_1} & V_{k_1+1,k_1+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ V_{k_n,k_1} & V_{k_n,k_1+1} & V_{k_n,k_1+2} & \cdots & V_{k_n,k_n-1} \end{pmatrix} \tag{14}$$
equals $k_n - k_1 - n + 1$, where
$$\begin{aligned} V_{k,k} &= k^{\underline{n}} - \sum_{s=1}^n k^{\underline{n-s}} \gamma_{n-s+1,-s}, \\ V_{k,j} &= -\sum_{s=1}^n j^{\underline{n-s}} \gamma_{n-s+1,k-j-s}, \quad j \leq k-1. \end{aligned}$$

Then the system (3) is locally analytically linearizable at the origin and, moreover, its linear representation can be found in the following *driftless form*

$$\dot{z} = g(t)u, \tag{15}$$

where the components of the vector function $g(t) = (g_1(t), \dots, g_n(t))^\top$ can be found as n linearly independent real analytic solutions of the differential equation

$$w^{(n)} = \sum_{k=1}^{n} \gamma_k(t) w^{(k-1)}, \tag{16}$$

where $w^{(j)}$ denotes the j-th derivative in t.

Remark 1: A change of variables $F(t,x)$ that reduces the system (3) to the driftless form (15) satisfies the following partial differential equations

$$F_t(t,x) + F_x(t,x)a(t,x) = 0,$$
$$F_x(t,x)\mathcal{R}^k b(t,x) = g^{(k)}(t), \quad k \geq 0.$$

However, it is more convenient to find it as a solution of the system

$$\begin{aligned} F_x(t,x) &= G(t)R^{-1}(t,x), \\ F_t(t,x) &= -F_x(t,x)a(t,x), \end{aligned} \tag{17}$$

where $G(t) = (g(t), \dot{g}(t), \dots, g^{(n-1)}(t))$; see also Remark 5 below.

Remark 2: Conditions 1 and 2 of Theorem 2 are analogous to linearizability conditions for autonomous systems as well as the requirements $\mathcal{R}^i b(t,x) \in C^1([0,\beta] \times Q)$, $i = 0, \dots, n$ [7]. Conditions 3 and 4 are specific for non-autonomous case [8], [15]. Condition $a(t,x) \in C^2([0,\beta] \times Q)$ is of technical character, see [14] for a detailed discussion.

Remark 3: Notice that any linear system (5) can be reduced to a driftless form by a non-autonomous change of variables $z = \Phi^{-1}(t)x$, where the matrix $\Phi(t)$ is defined as a nontrivial solution of the differential equation $\dot{\Phi} = A(t)\Phi$. In this case $g(t) = \Phi^{-1}(t)b(t)$. In particular, an autonomous linear system

$$\dot{x} = Ax + bu$$

is reduced to the driftless form (15) with $g(t) = e^{-At}b$ by the non-autonomous change of variables $z = e^{-At}x$.

Remark 4: If it is impossible to find an explicit solution of the differential equation (16), one can find sufficiently many coefficients of the Taylor series for a solution using the recurrent formula

$$w_k = -\frac{1}{V_{k,k}} \sum_{j=0}^{k-1} V_{k,j} w_j, \quad k \geq 0, \; k \neq k_i, \; i = 1, \dots, n, \tag{18}$$

where $w_{k_1}, \dots, w_{k_n}$ are arbitrary.

It is convenient to choose $g_i(t)$ such that $g_i(t) = -t^{k_i} + o(t^{k_n})$; in this case $L = I$. When using (18), one should choose $w_{k_i} = -1$ and $w_{k_j} = 0$ for $j \neq i$.

IV. Main result

Now we combine the theorems formulated in the previous section and present our main result.

Theorem 3: Suppose that the system (3) satisfies the conditions of Theorem 2 and, additionally,

$$V_{\ell,k_i} = 0 \quad \text{for} \quad \ell = k_i+1, \dots, k_n, \;\; i = 1, \dots, n-1. \tag{19}$$

Then there exist $\delta > 0$, a neighborhood $U(0)$ of the origin, and a (locally invertible) map $z = \widehat{F}(x) \in C^2(U(0))$, $\widehat{F}(0) = 0$, such that for any $x^0 \in U(0)$ the solution $(\widehat{\theta}(x^0), \widehat{u}(t; x^0))$ of the time-optimal control problem

$$\dot{x} = a(t,x) + b(t,x)u,$$
$$x(0) = x^0, \; x(\theta) = 0, \; |u(t)| \leq 1, \; \theta \to \min$$

can be found by the method of successive approximations as (13), where

$$\begin{aligned} y^0 &= \widehat{F}(x^0), \\ y^{r+1} &= \widehat{F}(x^0) + \int_0^{\theta(y^r)} g(t)u(t; y^r)dt + y^r \end{aligned} \tag{20}$$

for $r \geq 0$. Here components of $g(t)$ are n linearly independent analytic solutions of the differential equation (16) such that $g_i(t) = -t^{k_i} + o(t^{k_n})$, $i = 1, \dots, n$.

Proof: We notice that the condition (19) obviously implies that the rank of the matrix (14) equals $k_n - k_1 - n + 1$.

To prove the theorem, it is sufficient to show that (19) implies (12). However, it easily follows from the recurrent formula (18). Clearly, $\widehat{F}(x) = F(0,x)$, where $F(t,x)$ satisfies the system of differential equations (17) and $F(0,0) = 0$. ■

Remark 5: To apply Theorem 3, it is not necessary to solve the system (17). In fact, we only need to find $F(0,x^0)$, so we can proceed as follows. Denote

$$M(x) = G(t)R^{-1}(t,x)|_{t=0}$$

and, for any $k = 1, \dots, n$, consider the equations

$$\frac{\partial F_k(0,x)}{\partial x_s} = M_{ks}(x), \quad s = 1, \dots, n.$$

Successively for $s = 1, \dots, n$, solve (at least, numerically) the Cauchy problem for a single ordinary differential equation

$$z'(\tau) = M_{ks}(0, x_1^0, \dots, x_{s-1}^0, \tau, 0, \dots, 0),$$
$$z(0) = F_k(0, x_1^0, \dots, x_{s-1}^0, 0, \dots, 0),$$

then $F_k(0, x_1^0, \dots, x_{s-1}^0, x_s^0, 0, \dots, 0) = z(x_s^0)$. We find $F_k(0, x^0)$ after n such steps.

V. Example

As an illustrative example, we consider the following system

$$\begin{aligned} \dot{x}_1 &= u, \\ \dot{x}_2 &= (t - \tfrac{1}{4}t^4 + x_1x_3 + t^2x_1^2(1 + \tfrac{1}{2}\sin t))u, \\ \dot{x}_3 &= t^2(\sin t - 1)u - 2tx_1. \end{aligned} \tag{21}$$

First, we verify all the conditions of Theorem 3. Below we omit intermediate calculations (they can be done by hands or with any symbolic math package) and give the formulas for $\gamma_i(t)$, k_i and $g_i(t)$ only. We have $\det R(t,x) \neq 0$ for $t \in (0,3)$ and

$$\begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \gamma_3(t) \end{pmatrix} = R^{-1}(t,x)\mathcal{R}^3 b(t,x) =$$

$$= \begin{pmatrix} 0 \\ -3t \cdot \frac{(4t^2+4)\sin t + (t^3+2t)\cos t}{(t^5+4t^3-t^2+2)\sin t + (4t-t^4)\cos t} \\ \frac{(6t^4+12t^2-6t)\sin t + (t^5-t^2+6)\cos t}{(t^5+4t^3-t^2+2)\sin t + (4t-t^4)\cos t} \end{pmatrix}.$$

Therefore, $\gamma_1(t) = 0$, $\gamma_2(t) = -3t + O(t^3)$, $\gamma_3(t) = \frac{1}{t} + O(t)$ and the indicial equation takes the form

$$k(k-1)(k-2) - k(k-1) = 0$$

and has the roots $k_1 = 0$, $k_2 = 1$, $k_3 = 3$. Hence, the system (21) is locally linearizable and the time-optimal control problem for this system can be solved by the method of successive approximations. We find a system after linearization in the driftless form (15), where the components of $g(t)$ are solutions of the differential equation

$$w''' = \gamma_1(t)w + \gamma_2(t)w' + \gamma_3(t)w''.$$

It can be easily checked that

$$g_1(t) = -1,\ g_2(t) = -t + \tfrac{1}{4}t^4,\ g_3(t) = -t^2 \sin t$$

are its three linearly independent solutions. Hence, $L = I$ and in the new coordinates the system takes the driftless form

$$\dot{z}_1 = -u, \quad \dot{z}_2 = -(t - \tfrac{1}{4}t^4)u, \quad \dot{z}_3 = -t^2 \sin t\ u. \qquad (22)$$

We mention that even if we did not guess the solutions, we might find as many terms from their Taylor series as we want. Numerical calculations described below show that an approximation $g_3(t) \approx -t^3 + \frac{1}{6}t^5 - \frac{1}{120}t^7$ is quite enough for a reasonably accurate answer.

In this case, the power Markov moment min-problem (10) is of the form

$$y_1 = \int_0^\theta u(t)dt,\ y_2 = \int_0^\theta t\,u(t)dt,\ y_3 = \int_0^\theta t^3 u(t)dt,$$
$$|u(t)| \le 1,\ \theta \to \min. \qquad (23)$$

Its solution can be found directly. In fact, the optimal control is unique and equals ± 1 and has no more than two switchings. Since the set of points for which it has *less* than two switchings is of zero measure, we restrict ourselves by the case when there are *exactly* two switchings; denote them by t_1 and t_2. Let θ be the optimal time. Then

$$\pm y_1 = 2t_1 - 2t_2 + \theta,$$
$$\pm y_2 = t_1^2 - t_2^2 + \tfrac{1}{2}\theta^2,$$
$$\pm y_3 = \tfrac{1}{2}t_1^4 - \tfrac{1}{2}t_2^4 + \tfrac{1}{4}\theta^4,$$

where the upper (resp., lower) sign means that $u(t)$ equals $+1$ (resp, -1) on the first and the third intervals of constancy. Let us denote

$$c_1^\pm = \tfrac{1}{2}(\pm y_1 - \theta),\ c_2^\pm = \pm y_2 - \tfrac{1}{2}\theta^2,\ c_3 = 2(\pm y_3 - \tfrac{1}{4}\theta^4),$$

then

$$t_1 - t_2 = c_1^\pm, \quad t_1^2 - t_2^2 = c_2^\pm, \quad t_1^4 - t_2^4 = c_3^\pm.$$

Excluding t_1 and t_2, we get two equations w.r.t. θ

$$2c_3^\pm (c_1^\pm)^2 = (c_2^\pm)^3 + c_2^\pm (c_1^\pm)^4; \qquad (24)$$

they are polynomial equations in θ of degree 6. It can be shown that the optimal time equals the maximum of the roots of the equations (24); if this is a root of the equation corresponding to the upper (resp., lower) sign, then the optimal control equals $+1$ (resp, -1) on the first and the third intervals of constancy.

Finally, let us find $\widehat{F}(x^0)$. We have

$$\widehat{F}_x(x) = G(t)R^{-1}(t,x)|_{t=0} = \begin{pmatrix} -1 & 0 & 0 \\ x_1x_3 & -1 & \frac{1}{2}x_1^2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Obviously,

$$\widehat{F}_1(x) = -x_1, \quad \widehat{F}_3(x) = -x_3.$$

For $\widehat{F}_2(x^0)$, we follow the method described in Remark 5. Since $M_{2,1}(0) = 0$, we get $\widehat{F}_2(x_1^0, 0, 0) = 0$. Then,

$$\widehat{F}_2(x_1^0, x_2^0, 0) = -\int_0^{x_2^0} d\tau = -x_2^0.$$

Finally,

$$\widehat{F}_2(x_1^0, x_2^0, x_3^0) = -x_2^0 + \int_0^{x_3^0} \tfrac{1}{2}(x_1^0)^2 \tau d\tau = -x_2^0 + \tfrac{1}{2}(x_1^0)^2 x_3^0.$$

It can be shown that

$$F_1(t,x) = -x_1,\ F_2(t,x) = -x_2 + \tfrac{1}{2}x_1^2 x_3 + \tfrac{1}{2}x_1^3 t^2,$$
$$F_3(t,x) = -x_3 - x_1 t^2$$

but actually we do not need the form of $F(t,x)$ for the method described.

Suppose we solve the time-optimal control problem for the system (21) from the point $x^0 = (-0.5, -0.5, -0.5)$, then $y^0 = F(0, x^0) = (0.5, 0.4375, 0.5)$. Using the method of successive approximations described above, after 12 steps we achieve $\|y^{12} - y^{11}\| < 10^{-8}$, where $y^{12} \approx (0.5, 0.592, 0.603)$ and $t_1 \approx 0.0714$, $t_2 \approx 0.4496$, $\theta \approx 1.2564$; the trajectory components are shown in Fig. 1.

One can show that the functions $\{1, t - \frac{1}{4}t^4, t^2 \sin t\}$ form a Chebyshev system at least on the interval $[0, 2.2]$ (this is sufficient for this starting point and for the starting point that is considered below). As is mentioned above, this implies that the obtained control is time-optimal for the problem (22). Since it takes the point x^0 to the origin for the system (21), it is time-optimal for the initial problem.

If the starting point for the initial system is $x^0 = (-0.5, 0.5, -0.5)$, we get $y^0 = (0.5, -0.5625, 0.5)$ and the method of successive approximations diverges. However, one can apply the following modification: instead of (20), use the formula

$$y^0 = \widehat{F}(x^0),$$
$$y^{r+1} = \lambda\left(\widehat{F}(x^0) + \int_0^{\theta(y^r)} g(t)u(t; y^r)dt\right) + y^r, \quad r \ge 0,$$

where $\lambda \in (0, 1)$. One can show that the mapping leading to this recursive formula, i.e.,

$$y \mapsto \quad \lambda\left(\widehat{F}(x^0) + \int_0^{\theta(y)} g(t)u(t; y)dt\right) + y,$$

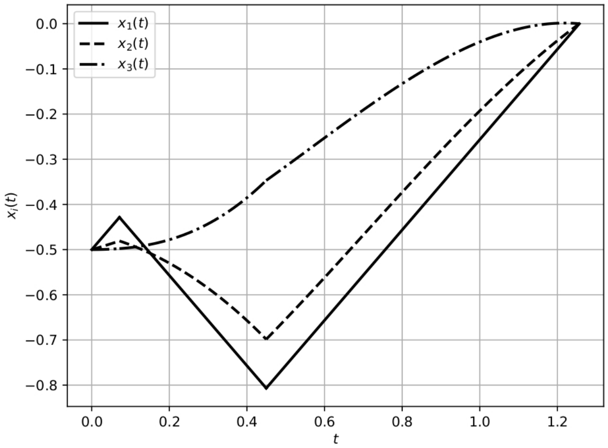

Fig. 1. Components of the optimal trajectory of the system (21) for $x^0 = (-0.5, -0.5, -0.5)$.

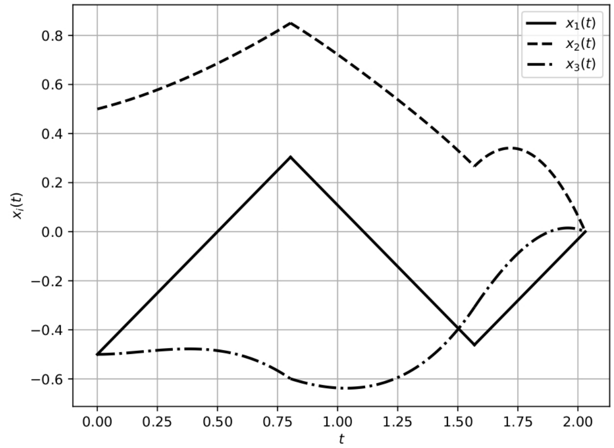

Fig. 2. Components of the optimal trajectory of the system (21) for $x^0 = (-0.5, 0.5, -0.5)$

is also a contraction. Though the contraction constant is greater, a domain where the method converges can be wider. So, in the previous example, if $\lambda = \frac{1}{2}$, then after 31 steps one has $\|y^{31} - y^{30}\| < 10^{-8}$, where $y^{31} \approx (0.5, 0.245, 1.427)$ and $t_1 \approx 0.8040$, $t_2 \approx 1.5693$, $\theta \approx 2.0305$; the trajectory components are shown in Fig. 2.

VI. Conclusion

The linearizability conditions, which allow us to determine whether a nonlinear control system can be mapped to a linear one, are well known. The linearizability property can be used, for example, to find an explicit solution of the time-optimal control problem. However, not all linear systems possess efficient methods for finding the time-optimal controls. Even if we know that the optimal control is bang-bang with $n-1$ switchings, the problem reduces to a very complex system of n nonlinear equations with respect to n variables.

In the paper, we propose conditions for a nonlinear non-autonomous control-affine system under which the system is locally analytically linearizable at the origin, and the corresponding linear non-autonomous system with real analytic coefficients admits an efficient method for solving the time-optimal control problem in a neighborhood of the origin. Namely, it is a method of successive approximations that involves solving a power Markov moment min-problem (possibly with gaps). Our approach does not require finding a linearizing change of variables. We provide an illustrative three-dimensional example to demonstrate the applicability of our method.

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