Boundary error-feedback regulation of a Sturm-Liouville dynamical system under distributed disturbances

Ilyasse Aksikas

Abstract— **This research addresses the regulator problem in a boundary control system that is generated by Sturm-Liouville dynamics. The primary purpose is to ensure the tracking of a desired reference trajectory while compensating for disturbances originating from a distributed parameter exosystem. At the start, a stabilizing state feedback regulator is constructed by employing a series representation based on the eigenvectors of the Sturm-Liouville generator, guiding the system output to align with the reference trajectory. Following this, a dynamic error-feedback regulator is formulated, utilizing the tracking error as the control input. In addition, the closed-loop system achieves exponential stability, with the tracking error converging to zero over time.**

Index Terms— **Regulator problems, Sturm-Liouville system, Sylvester equation, infinite-dimensional systems.**

I. INTRODUCTION

The behavior of many physical systems—such as those involving vibrations, diffusion processes, heat transfer, fluid flow, wave propagation, and chemical reactions with axial dispersion—is frequently governed by Sturm-Liouville (SL) operators (see, for instance, [27], [25]). These operators are essential in modeling how such systems evolve over time and under varying conditions. A substantial body of theoretical research on SL operators and their properties is available in the literature (see [24], [7], [29], [27], [25]). Notably, SL operators are central to spectral theory, which examines the eigenvalues and eigenfunctions associated with differential equations. The eigenfunctions of these operators generally form an orthogonal set with respect to a defined inner product. This orthogonality is a key feature that allows complex problems to be broken down into simpler components. Using this property facilitates the representation and analysis of solutions to boundary value problems in terms of eigenfunctions. In this study, a spectral strategy is implemented to address the regulation problem for a Sturm-Liouville boundary control system (SLBCS) subjected to disturbances, as modeled by the following partial differential equation (PDEs).

$$\begin{cases} z_t(t,\xi) &= \dfrac{1}{\rho(\xi)}\Big((p(\xi)z_\xi(t,\xi))_\xi + q(\xi)z\Big) \\ & \qquad\qquad +f(\xi)\upsilon(t,\xi) \\ y(t) &= \displaystyle\int_\alpha^\beta \rho(\xi)c(\xi)z(t,\xi)d\xi \end{cases} \tag{1}$$

subject to the following boundary and initial constraints

$$\begin{cases} \alpha_1 z_\xi(t,\alpha) + \alpha_2 z(t,\alpha) = \nu(t) \\ \beta_1 z_\xi(t,\beta) + \beta_2 z(t,\beta) = 0 \\ z(0,\xi) = z_0(\xi) \end{cases} \tag{2}$$

where z represents the system's state, ν represent the input and υ represents disturbance signals. The constants $\alpha_1, \alpha_2, \beta_1$ and β_2 are real numbers such that $(\alpha_1, \alpha_2) \neq (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$ and ρ, p, $\dfrac{dp}{d\xi}$, q and f are real-valued functions with $\rho >$ and $p > 0$.

Control of PDEs presents unique challenges that differ fundamentally from those encountered in the control of ordinary differential equations (ODEs). A key factor behind these challenges is the infinite-dimensional nature of PDEs, which adds significant complexity to the analysis and design of control strategies. Unlike ordinary differential equations (ODEs), which model systems with a finite number of state variables evolving over time and are therefore confined to finite-dimensional spaces, partial differential equations (PDEs) govern the dynamics of systems whose states vary continuously over both time and space. This spatial distribution of state variables across an infinite-dimensional domain introduces additional layers of complexity in modeling, analysis, and control design, making the control of PDE-governed systems inherently more intricate than that of their ODE counterparts. To address the distributed characteristics of PDE systems, an infinite-dimensional state-space framework is essential. This approach models the system within an abstract space, treating it as a differential equation operating in an infinite-dimensional setting [12], [6], [5]. By doing so, the framework enables the analysis of PDE systems with methods analogous to those used for ODEs, with the additional complexity introduced by their infinite-dimensional nature. Effectively employing this approach requires advanced mathematical tools, particularly functional analysis and operator theory. Several methodologies are employed in the control of PDEs, each addressing different aspects of the control problem. Optimal control techniques ([6], [22]) focus on finding control laws that optimize a performance criterion while satisfying system constraints. Sliding mode control ([19]) offers robustness against disturbances and model uncertainties by designing control laws that drive the system's trajectory to a predefined sliding surface. Robust control ([10], [11]) aims to ensure system performance and stability despite uncertainties and variations in system parameters. Model predictive control ([15], [20]) involves solving an optimization problem at each time step

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to predict and control the future behavior of the system based on a model. Backstepping control ([8]) is a design methodology that systematically stabilizes PDE systems by transforming them into simpler forms. Observer design ([1], [2], [21]) involves creating estimators to infer unmeasured states from available measurements, which is crucial for feedback control. Disturbances in dynamical systems refer to external factors or variations that influence the system's behavior and performance over time. These disturbances can manifest as sudden shocks, random noise, fluctuations in input parameters, or changes in environmental conditions. Such disruptions can compromise the stability and reliability of a system, making it critical to analyze their effects and devise strategies to mitigate them. This is particularly significant in areas like control theory, engineering, and physics, where maintaining the stability and robustness of systems is vital for their efficient operation. To address disturbances and sustain optimal performance under varying conditions, researchers use various control strategies, including feedback control, filtering techniques, and adaptive algorithms. One prominent approach to managing disturbances is to solve the servomechanism, or regulator, problem. This consists of constructing a feedback regulator that can track a desired output while rejecting disturbances and ensuring the stability of the closed-loop system. The servomechanism problem typically includes an exosystem that generates the reference trajectory and disturbances, both of which the regulator must handle effectively (see [13], [17]). Within this framework, there are two primary variants of the regulator problem. The state feedback regulator relies on full access to the trajectories of both the system and the exosystem, offering a straightforward design process but limited practicality due to the difficulty of acquiring complete trajectory information in real-world applications. Alternatively, the error-feedback regulator employs the tracking error, defined as the difference between the desired and actual output, as the control basis. This approach is more practical, focusing on minimizing the tracking error rather than requiring complete trajectory data. In the context of ordinary differential equation (ODE) systems, regulation problems have been extensively studied, with significant progress achieved using the internal model principle (IMP). The IMP provides a robust framework for designing controllers that address both desired outputs and disturbances by embedding models of the exosystem within the control structure (see [13], [16], [17]). This principle has been pivotal in the advancement of feedback control theory and its applications in managing disturbances across a wide range of systems. Significant progress has been made in extending output regulation results to infinite-dimensional systems. For instance, [9] expands the geometric framework to include linear distributed parameter systems with bounded input and output operators, addressing scenarios where reference signals and disturbances are generated by a finite-dimensional exosystem. The study provides clear criteria for solving regulator equations, relying on the eigenvalues of the exosystem and the system's transfer function. Subsequent research, such as in [23], expanded these results to the unbounded case, driven by interest in systems where control and/or observation operators are unbounded. Regulation for infinite-dimensional exosystems is studied in [18], showing that a feedback controller incorporating an internal model can regulate signals from the exosystem while achieving strong or weak stabilization of the closed-loop system. Likewise, [4] investigates output regulation for a reverse flow reactor modeled by a hyperbolic PDE system, employing the operator Riccati equation to design stabilizing feedback and output injection gains. Finally, the spectral approach has been employed to address the regulation problem for a catalytic cracking process, as detailed in [3].

This study focuses on exploring the use of existing research related to the regulator problem, with a special emphasis on applying the spectral approach to control the Sturm-Liouville boundary control system (SLBCS) as described by equations (1)-(2). The focus is on leveraging the spectral approach, a well-established method recognized for its effectiveness in addressing complex control challenges, to enhance the regulation of the SLBCS.

II. State-space description

The Sturm-Liouville boundary control system (SLBCS), along with its associated output function, can be expressed as a linear system operating within the framework of a Hilbert space. $L^2(\alpha,\beta)$

$$\left\{\begin{array}{lcl} \dot z(t) &=& \mathcal{A}z(t)+\mathcal{F}\upsilon(t),\ z(0)=z_0\\ \mathcal{B}z(t) &=& \nu(t)\\ y(t) &=& \mathcal{C}x(t)\end{array}\right. \tag{3}$$

where the operator $\mathcal{A}$ is defined for every function $g\in L^2(\alpha,\beta)$ such that g and $\frac{dg}{d\xi}$ are absolutely continuous and $\frac{d^2g}{d\xi^2}\in L^2(\alpha,\beta)$ and satisfies $\beta_1\frac{dg}{d\xi}(b)+\beta_2 g(b)=0$.

$$\mathcal{A}g=\frac{1}{\rho(\xi)}\left(\frac{d}{d\xi}\left(p(\xi)\frac{dg}{d\xi}(\xi)\right)+q(\xi)g(\xi)\right) \tag{4}$$

Denote by $D(\mathcal{A})$ the domain of definition of $\mathcal{A}$. The input operator $\mathcal{B}:L^2(\alpha,\beta)\to\mathbb{R}$ is given by

$$h\ \mapsto\ \mathcal{B}g=\left[\alpha_1\frac{d\cdot}{d\xi}+\alpha_2 I\right]_{|\xi=\alpha}h \tag{5}$$

The operator $\mathcal{F}:L^2(\alpha,\beta)\to L^2(\alpha,\beta)$ is given by $\mathcal{F}=f(\xi)\cdot I$ and the operator $\mathcal{C}:L^2(\alpha,\beta)\mapsto\mathbb{R}$ is given by $\mathcal{C}x=\langle c,x\rangle_\rho$. As stated in [14], the operator $\mathcal{A}$, defined in (4), exhibits the following key properties: (i) It is a Riesz spectral operator and generator of a C_0-semigroup on the space $L^2(\alpha,\beta)$, and (ii) Its eigenvalues $\{\lambda_n\}_{n\geq 1}$are simple, real and countable. Furthermore, the associated set of normalized eigenfunctions $\theta_n: n\geq 1$ constitutes an orthonormal basis regarding the inner product $\langle\cdot,\cdot\rangle_\rho$., i.e.

$$\langle\theta_n,\theta_m\rangle_\rho=\int_\alpha^\beta\rho(\xi)\theta_n(\xi)\theta_m(\xi)d\xi=0,\quad\text{for}\quad n\neq m$$

In the subsequent analysis, for any function $g\in L^2(\alpha,\beta)$, we define $g_n:=\langle g,\theta_n\rangle_\rho$. Denote by $\mathbb{H}=\mathbb{R}\oplus L^2(a,b)$

the Cartesian product of $L^2(\alpha,\beta)$ and $\mathbb{R}$ with the inner product and norm regarding the weight function ρ for all $\mathbf{g}=(\nu_1,g)^T, \mathbf{h}=(\nu_2,h)^T \in \mathbb{H}$.

$$\langle \mathbf{g},\mathbf{h}\rangle_{\mathbb{H}} = \nu_1\nu_2 + \langle g,h\rangle_\rho \qquad \text{and} \qquad \|\mathbf{g}\|_{\mathbb{H}} = \sqrt{\nu_1^2 + \|g\|_\rho^2}$$

By applying the established procedure detailed in [12, Section 3.3], it can be shown that when $B_0 = d(\xi)\cdot I$ such that $d \in D(\mathcal{A})$ and satisfies the following condition

$$\alpha_1 d'(\alpha) + \alpha_2 d(\alpha) = 1 \tag{6}$$

then the SLBCS (3) can be transformed into the following state-space description on $\mathbb{H}$:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + F\upsilon(t) \qquad x(0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{7}$$

where the new trajectories are

$$x(t) := \begin{bmatrix} \nu(t) \\ z(t) - B_0\nu(t) \end{bmatrix} \text{ and } \quad u(t) = \dot{\nu}(t) \tag{8}$$

and the new generators are

$$A = \begin{bmatrix} 0 & 0 \\ hI & A_0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ -dI \end{bmatrix},$$

$$F = \begin{bmatrix} 0 \\ \mathcal{F} \end{bmatrix} \text{ and } C = [\mathcal{C}B_0 \; \mathcal{C}] \, .$$

such that $A_0 : D(A_0) \to L^2(\alpha,\beta)$ is defined as the restriction of $\mathcal{A}$ to the kernel of $\mathcal{B}$, with its domain given by $D(A_0) = D(\mathcal{A}) \cap Ker(\mathcal{B})$. Additionally, the function h is expressed by

$$h(\xi) = \frac{1}{\rho(\xi)}\Big((p(\xi)d'(\xi))' + q(\xi)d(\xi) \Big) = \mathcal{A}d(\xi) \tag{9}$$

It can be demonstrated, using integration by parts, that the operator A_0 is self-adjoint ($A_0^* = A_0$). Furthermore, observe that the adjoints operators of B_0 and $\mathcal{C}$ are:

$$B_0^* z = \langle d, z\rangle_\rho \quad \forall z \in H \qquad \text{and} \qquad \mathcal{C}^* y = cy, \quad \forall y \in \mathbb{R}.$$

which implies that $\forall \nu, y \in \mathbb{R},\ \forall z \in L^2(\alpha,\beta)$

$$B^* \begin{bmatrix} \nu \\ z \end{bmatrix} = \nu - \langle d, z\rangle_\rho \quad \text{ and } C^* y = \begin{bmatrix} \langle d, c\rangle_\rho y \\ cy \end{bmatrix}.$$

Observe that the spectrum of the operator A is $\sigma(A) = \sigma(\mathcal{A}) \cup \{0\}$ and the corresponding eigenvectors are:

$$\Phi_0 = \begin{pmatrix} 1 \\ -d \end{pmatrix} \in \mathbb{H}, \quad \text{and} \quad \Phi_n = \begin{pmatrix} 0 \\ \theta_n \end{pmatrix} \in \mathbb{H},\ n \geq 1 \tag{10}$$

and the eigenvectors of A^* are:

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{H} \quad \text{and} \quad \Psi_n = \begin{pmatrix} \lambda_n^{-1}h_n \\ \theta_n \end{pmatrix} \in \mathbb{H}, \quad n \geq 1 \tag{11}$$

Furthermore, the bio-orthogonality property can be easily validated, i.e, $\langle \Phi_n, \Psi_m\rangle_{\mathbb{H}} = \delta_{nm}$.

III. State-feedback regulator design

This section investigates the state-feedback regulator problem, a crucial step in addressing the error-feedback regulator problem. It involves designing a controller that takes advantage of the full state information of the system to meet specified performance goals. Its primary role is to drive the system's behavior, ensuring that it reacts effectively to disturbances. We assume that both the disturbance and reference signals are ge by

$$\dot{\upsilon}(t) = S\upsilon(t), \quad \upsilon(0) = \upsilon_0 \in L^2(\alpha,\beta) \tag{12}$$

such that $S : D(S) \subset L^2(\alpha,\beta) \to L^2(\alpha,\beta)$ is an operator expressed as:

$$S\upsilon = \sum_{n=1}^{\infty} s_n \langle \upsilon, \theta_n\rangle_\rho \theta_n = \sum_{n=1}^{\infty} s_n \upsilon_n \theta_n \tag{13}$$

where $s_n \in i\mathbb{R}$, $\lim_{n\to\infty} Im(s_n) = \pm\infty$ and $s_n \neq s_k$, for $n \neq k$. The domain of S is $D(S) = \{\upsilon \in L^2(\alpha,\beta) \mid \sum_{n=1}^{\infty} |s_n|^2|\upsilon_n|^2 < \infty\}$. The targeted output is given by

$$y_r(t) = Q\upsilon(t) := \langle q, \upsilon\rangle_\rho = \int_\alpha^\beta \rho(\xi)q(\xi)\upsilon(\xi,t)d\xi$$

with q is a continuous on $[\alpha,\beta]$. Here, we want to design a state and disturbance feedback regulator of the form $u(t) = Kx(t) + L\upsilon(t)$, where the operators K and L are bounded. These operators are designed such that (i) $A+BK$ generates an exponentially stable trajectory and (ii) the error $e(t) = y(t) - y_r(t)$ converges asymptotically to zero. Let us begin by concentrating on the design of a stabilizing feedback K. Building on [12, Theorem 5.1.3], the following result demonstrates that a stabilizing feedback can be achieved by solving a set of infinite algebraic Riccati equations.

Theorem 1: A stabilizing feedback operator K of the SL augmented system described by (7) is expressed as follows

$$\begin{aligned} Kx &= \sum_{m=0}^{\infty} K_m \langle x, \Psi_m\rangle_{\mathbb{H}} \\ &:= \sum_{m=0}^{\infty} \left(-\Gamma_{0m} + \sum_{n=1}^{\infty} \epsilon_n \Gamma_{nm} \right) \langle x, \Psi_m\rangle_{\mathbb{H}} \end{aligned} \tag{14}$$

where $\epsilon_n = l_n - \lambda_n^{-1} g_n$ and such that $\Gamma_{nm} = \Gamma_{mn}$ satisfy the set of algebraic equations below for all $n, m \geq 0$

$$(\lambda_n + \lambda_m)\Gamma_{nm} + N_{nm} - 2\sum_{k,l=0}^{\infty} \Gamma_{nk}\Gamma_{ml}B_{kl} = 0, \tag{15}$$

and N_{nm} represent the components of a given positive definite operator N on the basis $\{\Phi_n\}$, i.e $N_{nm} = \langle N\Phi_n, \Phi_m\rangle_{\mathbb{H}}$. Moreover, a stabilizing boundary control law of the system (3) in the absence of disturbances can be expanded using the following series

$$\nu(t) = \sum_{m=1}^{\infty} e^{-\Lambda t} \int_0^t e^{\Lambda\tau} K_m z_m(\tau) d\tau \tag{16}$$

where $\Lambda = -K_o + \sum_{m=1}^{\infty} \epsilon_m K_m$.

Proof: The idea is to use the associated Lyapunov equation together with the spectral approach to get (15). Moreover, expansion (16) can be obtained by rewriting (14) in terms of the original variables. ■

To obtain a specific stabilizing feedback, we can introduce further constraints on the design operator N to streamline Equation (15). For example, specifying a particular structure for N can simplify the process of solving the algebraic Riccati equation, making it easier to identify a stabilizing feedback that is efficient and practical.

Corollary 1: Suppose that the design operator N is selected such that $N_{nm} = 0$, for $n \neq m$. Then, a stabilizing feedback operator K of system (7) is expressed by (14) such that the constants K_m are explicitly given for all $m \geq 1$

$$K_0 = -\frac{N_{00}}{2} \quad \text{and} \quad K_m = \frac{\lambda_m + \sqrt{\lambda_m^2 + 2N_{mm}\epsilon_m^2}}{2\epsilon_m}. \tag{17}$$

Moreover, a stabilizing boundary control law of the system (3) in the absence of disturbances is given by (16) where Λ is explicitly given by

$$\Lambda = \left(\frac{N_{00}}{2} + \sum_{n=1}^{\infty} \frac{\lambda_n + \sqrt{\lambda_n^2 + 2N_{nn}\epsilon_n^2}}{2} \right). \tag{18}$$

We now delve into the problem by considering the presence of disturbances; more specifically, we want to design a disturbance-feedback controller L. This operator plays a pivotal role in mitigating the impact of disturbances by adapting the feedback control mechanism appropriately. To define and determine the structure of the disturbance-feedback operator L, we leverage a key lemma that establishes the theoretical groundwork. This lemma enables the derivation of L by solving a Sylvester operator equation. The proof of the lemma adopts a methodology analogous to that used for Lemma 1 in [4].

Lemma 1: Let us consider the augmented SL system given by (7). Provided that there are two operators $X \in \mathcal{L}(L^2(\alpha,\beta),\mathbb{H})$ and $T \in \mathcal{L}(L^2(\alpha,\beta),\mathbb{R})$ that satisfy the following constrained Sylvester equation

$$AX - XS + BT + F = 0 \tag{19}$$

$$CX - Q = 0 \tag{20}$$

then the input $u(t) = Kx(t) + Lv(t)$, where K is specified by equation (14) and $L = T - KX$, ensures that the error $e(t)$ asymptotically approaches zero.

To take advantage of the generator spectrum A in addressing the Sylvester equation (19), the lemma below is indispensable. This lemma establishes that, under specific conditions, the solution to the constrained Sylvester equation can be expanded as a series involving the eigenvalues and eigenvectors of A. The proof employs an approach analogous to that outlined in [18, Lemma 6], providing a systematic framework for deriving the solution.

Lemma 2: Let $D \in \mathcal{L}(L^2(\alpha,\beta),\mathbb{H})$. Then the Sylvester equation

$$AX - XS + D = 0, \qquad XD(S) \subset D(A) \tag{21}$$

admits the unique solution

$$X = \sum_{n=1}^{\infty} \langle \cdot, \theta_n \rangle_\rho \mathcal{R}(s_n, A) D\theta_n \in \mathcal{L}(L^2(\alpha,\beta),\mathbb{H}) \tag{22}$$

if and only if

$$\sup_{\|z\|\leq 1} \left(\sum_{n=1}^{\infty} |\langle \mathcal{R}(s_n, A) D\theta_n, z \rangle_\rho|^2 \right)^{1/2} < \infty. \tag{23}$$

where $\mathcal{R}(s_n, A) := (s_n I - A)^{-1}$.

Next, we explore the Sylvester equations (19) and (20) in the context of the Sturm-Liouville (SL) system described by equation (7). To thoroughly analyze these equations and assess their significance for the SL system, it is essential to establish a set of foundational assumptions. These assumptions will streamline the mathematical formulation and ensure the existence and uniqueness of solutions, laying the groundwork for a rigorous examination of the problem.

(A1) $D := BT + F$ satisfies inequality (23), i.e

$$\sup_{\|z\|\leq 1} \left(\sum_{n=1}^{\infty} |\langle \mathcal{R}(s_n, A)(BT + F)\theta_n, z \rangle_\rho|^2 \right)^{1/2} < \infty$$

(A2) The operator $T : L^2(\alpha,\beta) \to \mathbb{R}$ has the form $Tv = \langle \gamma, v \rangle_\rho$ $\forall v \in L^2(\alpha,\beta)$, where $\gamma \in L^\infty(\alpha,\beta)$ is chosen such that the sequence $\gamma_n := \langle \gamma, \theta_n \rangle_\rho$ satisfies the following condition $\forall n \geq 1$

$$\sum_{m=1}^{\infty} X_{nm} \langle c, \theta_m \rangle_\rho = \langle q, \theta_n \rangle_\rho \tag{24}$$

Now, the core idea is to express the disturbance-feedback operator L as a series expansion of $\{\theta_n\}_{n\geq 1}$, using the spectral properties of the system. This approach ensures targeted disturbance compensation, maintaining stability by focusing on the most sensitive system modes, while utilizing the orthogonality and completeness of the eigenvectors for robust regulator performance.

Theorem 2: Consider the SLBCS described by (3) assuming the presence of disturbances arising from the exosystem (12). Suppose that assumptions (A1) and (A2) satisfied. Provided that K is given by (14) and L is given for all $v \in L^2(\alpha,\beta)$ by

$$Lv = \sum_{n=1}^{\infty} L_n \langle v, \theta_n \rangle_\rho \tag{25}$$

where L_n are given explicitly for $n \geq 1$ by

$$L_n = \gamma_n \left(1 + \frac{N_{00}}{2s_n} \right) + \sum_{m=1}^{\infty} \frac{\lambda_m + \sqrt{\lambda_m^2 + 2N_{mm}\epsilon_m^2}}{2(s_n - \lambda_m)} \left[\gamma_n - \frac{f_{nm}}{\epsilon_m} \right] \tag{26}$$

where $f_{nm} = \langle f\theta_n, \theta_m\rangle_\rho$. Then the state feedback controller drives the tracking error asymptotically toward zero.

$$u(t) = Kx(t) + L\upsilon(t). \tag{27}$$

Now, it becomes possible to determine the original stabilizing boundary input ν as feedback of the state z and the disturbance υ. To accomplish this, we will formulate and solve a linear differential equation.

Corollary 2: Consider the SLBCS described by (3) assuming the presence of disturbances arising from the exosystem (12). Suppose that (A1) and (A2) hold. Then the stablizing boundary input

$$\nu(t) = e^{\Lambda t}\int_0^t e^{-\Lambda t}\left(\sum_{n=1}^{\infty} K_n z_n(t) + \sum_{n=1}^{\infty} L_n \upsilon_n(t)\right) dt \tag{28}$$

guides the error $e(t)$ towards zero.

IV. Error-Feedback Regulator

In many practical situations, obtaining full access to the state and disturbance information—something that is essential for executing the state-feedback regulator described in equation (27)—is often not feasible. The limitations could be due to various factors such as the difficulty in measuring all relevant states, the presence of unobservable disturbances, or constraints in data acquisition and processing. As a result, there is a necessity to address the servomechanism problem. This problem incorporates devising a control strategy that can operate effectively even when full information about the state and disturbances is unavailable. The goal is to develop a dynamical controller that performs well in more practical scenarios and utilizes the tracking error as an input. The controller is taking the form

$$\begin{cases} \dot\varphi(t) &= \hat A\varphi(t) + \hat B e(t) \\ u(t) &= H\varphi(t) \end{cases} \tag{29}$$

provided that the operator $\mathfrak{A} = \begin{pmatrix} A & \hat B H \\ \hat B C & \hat A \end{pmatrix}$ generates an exponentially stable trajectory and the error $e(t) = y(t) - y_r(t)$ asymptotically converges to zero.

The dynamical controller (29) functions as a compensator, utilizing the error as input to generate the estimate of the system state and disturbance, represented by $\varphi(t) \simeq (x(t), \upsilon(t))^T \in \mathbb{H} \oplus L^2(\alpha,\beta) := \mathbb{H}$. The latter is equipped by the usual inner product and denoted by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$. The second equation of (29) embodies the feedback mechanism on the estimated state and disturbance. In the preceding section, a stabilizing feedback operator H was derived. Indeed, $H = [K \quad L]$, provided that the operators K and L are specified by (14) and (25), respectively. Consider the operators A_0 and C_0 given by

$$\hat A_0 = \begin{pmatrix} A & F \\ 0 & S \end{pmatrix} \quad \text{and} \quad \hat C_0 = (C \quad -Q)$$

where the domain of definition of $\hat A_0$ is given by $D(\hat A_0) = D(A) \oplus D(S)$. Let $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ be the output compensation operator associated with the pair $(\hat C_0, \hat A_0)$. As per [9, Theorem 4.2], if $\hat A$ is selected to be

$$\hat A = \begin{pmatrix} A + BK - G_1 C & F + BL + G_1 Q \\ -G_2 C & S + G_2 Q \end{pmatrix}$$

then the tracking error approaches zero when t approaches infinity. To complete the controller design (29), it is essential to identify the stabilizing compensation operator G. It is easy to notice that $\sigma(\hat A_0) = \sigma(A) \cup \sigma(S) = \{\zeta_n : \zeta_{2n} = \lambda_n, \zeta_{2n+1} = s_{n+1},\ n \geq 0\}$ with eigenvectors

$$\Omega_0 = \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -l \\ 0 \end{pmatrix}, \ \Omega_{2n} = \begin{pmatrix} \Phi_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta_n \\ 0 \end{pmatrix}$$

and

$$\Omega_{2n+1} = \begin{pmatrix} (s_{n+1} I - A)^{-1} F\theta_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \displaystyle\sum_{k=1}^{\infty} \frac{f_{n+1,k}}{s_{n+1} - \lambda_k}\theta_k \\ \theta_{n+1} \end{pmatrix}$$

and $\sigma(\hat A_0^*) = \sigma(A^*) \cup \sigma(S^*) = \{\pi_n : \pi_{2n} = \lambda_n, \pi_{2n+1} = \overline{s_{n+1}} = -s_{n+1},\ n \geq 0\}$ with eigenvectors

$$\Theta_0 = \begin{pmatrix} \Psi_0 \\ (\lambda_0 I - S^*)^{-1} F^* \Psi_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\Theta_{2n} = \begin{pmatrix} \Psi_n \\ (\lambda_n I - S^*)^{-1} F^* \Psi_n \end{pmatrix} = \begin{pmatrix} \lambda_n^{-1} g_n \\ \theta_n \\ \displaystyle\sum_{k=1}^{\infty} \frac{f_n\langle 1, \theta_k\rangle_\rho}{\lambda_n + s_k}\theta_k \end{pmatrix}$$

$$\text{and} \quad \Theta_{2n+1} = \begin{pmatrix} 0 \\ 0 \\ \theta_{n+1} \end{pmatrix}$$

The subsequent theorem affirms that deriving the output compensation operator G is achievable through solving a series of Riccati equations.

Theorem 3: Let $\hat M$ be a definite positive operator $\hat M$. The output compensation operator of the pair $(\hat C_0, \hat A_0)$ is described for all $y \in \mathbb{R}$ by

$$Gy = -\Pi \hat C_0^* y = -\sum_{n,m=0}^{\infty} \Pi_{nm}\langle \hat C_0^* y, \Omega_m\rangle_{\mathbb{H}} \Omega_n \tag{30}$$

provided that $\{\Pi_{nm}\}_{n,m\geq 0}$ satisfy

$$(\pi_n + \rho_m)\Pi_{nm} + \hat M_{nm} - 2\sum_{k,l=0}^{\infty} \Pi_{nk}\Pi_{ml}\hat C_{kl} = 0 \tag{31}$$

where $\hat M_{nm} = \langle \hat M\Theta_n, \Theta_m\rangle_{\mathbb{H}}$ and $\hat C_{nm} = \langle \hat C_0\Theta_m, \hat C_0\Theta_n\rangle_{\mathbb{H}}$. Similarly to what has been done for the stabilizing feedback operator, the output compensation operator can be found based on some assumptions about the design operator $\hat M$.

Corollary 3: Consider a definite positive design operator

$\hat{M}$ such that $\hat{M}_{nm} = 0$, for $n \neq m$. Then an output compensation operator for $\hat{C}_0, \hat{A}_0$ is expressed for $y \in \mathbf{R}$ by

$$Gy = -\sum_{n=0}^{\infty} \frac{\pi_n + \sqrt{\pi_n^2 + 2\hat{M}_{nn}\hat{C}_{nn}}}{2\hat{C}_{nn}} \langle \hat{C}_0^* y, \Omega_n \rangle_{\mathbb{H}} \Omega_n \quad (32)$$

V. CONCLUSIONS

This paper explores the servomechanism problem in the context of a Sturm-Liouville boundary control system that is influenced by disturbances from a distributed parameter exosystem. The core contribution of this study is the application of a spectral approach within an infinite-dimensional setting to develop feedback regulators. These regulators are formulated in terms of the eigenvalues and eigenvectors of the Sturm-Liouville (SL) generator, providing a robust framework for addressing the control problem.

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