

Robust Distributed Fractional-Order Dynamic Output Feedback for Limited-Time Consensus Control in Multi-Agent Systems

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Abstract—This paper presents a new approach to tackling one of the intricate challenges of consensus control in distributed systems, specifically targeting the stabilization of fractional-order outputs in linear fractional-order multi-agent systems. We introduce an innovative distributed feedback control strategy that leverages dynamic output feedback to stabilize the closed-loop system. By refining the H_2 robust control method, the controller and observer gains are precisely determined through an eigenvalue-based optimization process. The effectiveness and robustness of the proposed methodology are validated through simulations, with graphical results illustrating enhanced system performance and stability.

Keywords: consensus control, fractional order system, Dynamic Output feedback, stability.

I. INTRODUCTION

In recent decades, fractional order calculus has gained interest across scientific fields, particularly engineering, due to its ability to more accurately model systems with memory effects, such as heat transfer and glucose-insulin processes. These systems' future behavior is influenced by their past, making fractional order differential equations important for predictive modeling. Fractional order control methods provide improved performance over traditional controllers, including fractional order PID controllers [1], [2], [3], [4], [5], [6], which are applied in areas like unmanned aircraft [7], wind turbines [8], voltage regulators [9], chaotic systems [10], and chemical processes. Multi-agent systems, where multiple agents work together to achieve shared goals, have applications in robotics, spacecraft, satellite positioning, and intelligent power grids [11]. These systems also play a role in security, where leader-follower dynamics are modeled using mathematical games [12]. Consensus in multi-agent systems refers to agents reaching agreement over time through protocols, categorized into leader-less [13] and leader-follower [14] problems. Leader-follower systems involve one agent determining the group's path, while leaderless systems rely on agents making decisions based on information from their neighbors. Research on consensus problems often focuses on agents with correct order dynamics, such as first-order [15], second-order [16], and higher-order dynamics [17], and also addresses applications like blockchain systems with Byzantine nodes [18]. This problem is studied for linear [19], second-order [20], and nonlinear systems with external disturbances [21]. The discussed study explores robustness in consensus as an inherent property, particularly in time-limited systems, which offer increased accuracy and disturbance resistance. The consensus problem for fractional order multi-agent systems is studied using the Lyapunov theorem [22] and sliding mode control [23], showing that time-bound consensus can be achieved with directed spanning trees. The paper introduces finite-time [24] and fixed-time convergence [25], where consensus is reached within bounded time frames, regardless of initial conditions. Fixed-time consensus

design has been explored for first-order [26], [27], higher-order [28], and heterogeneous systems [29]. Additionally, the research discusses event-based adaptive fuzzy control [30] and methods for handling robot delays using non-linear cut-off times [31]. A new control law is introduced to address chattering in signals, along with a distributed protocol for consensus [32]. The research highlights key innovations. This paper assumes that each agent has a single integral dynamics of fractional order. It examines the issue of consensus for this type of system and presents a distributed protocol to achieve consensus. The research presents a new fractional-order distributed consensus rule aimed at achieving consensus with a constant time convergence rate for multi-agent fractional-order systems. In order to achieve leader-follower consensus in a leader-follower system, we first define a suitable performance output and then introduce an associated H_2 cost functional. The objective is to develop distributed protocols using fractional order dynamic output feedback control that enable the multi-agent system to reach consensus from initially distinct starting points, or also after a temporary distortion, while minimizing the associated H_2 cost functional. Due to communication constraints among the agents, this problem is non-convex and a closed-form solution has not been provided in the literature so far. Therefore, we are looking for an alternative approach that focuses on sub-optimality. The application of our strategy is in fractional-order multi-agent systems, where limited-time consensus ensures that the agents return to a stable operational (normal) state within a known or, at least, controllable time frame

II. PRELIMINARIES

A. Notation:

The symbols \mathbb{R} , \mathbb{R}^n , and $\mathbb{R}^{m \times n}$ represent the field of real numbers, n -dimensional real vector space, and $m \times n$ real matrices, respectively. The superscript T denotes transposition for matrices and vectors. I , This is the $n \times n$ identity matrix. The trace of a square matrix A is represented as $\text{tr}(A)$. If all eigenvalues of a matrix have negative real parts, the matrix is called Hurwitz (or stable). For a symmetric matrix P , we write $P < 0$ for negative definiteness and $P > 0$ for positive definiteness. A diagonally matrix of size $n \times n$ with (d_1, \dots, d_n) dimension, on the diagonal is represented by $\text{diag}(d_1, \dots, d_n)$. Let $\text{blockdiag}(M_1, \dots, M_m)$ be the block diagonal matrix with diagonal blocks M_i for matrices M_1, \dots, M_m . $A \otimes B$ represents the Kroneckerproduct of matrix A and matrix B .

B. Graph Theory:

$\mathcal{G} = (V, E)$ represents a directed graph, where $V = \{1, \dots, N\}$ and $E = \{e_1, \dots, e_M\}$, with E being a subset of $E \subset V \times V$. The pair $(i, j) \in E$ represents the edge from node i to node j . When $(i, j) \in E$ implies $(j, i) \in E$, we say the graph is undirected. When $(i, i) \notin E$ for all $i \in V$ which implies, i.e., without self-loops, we say the graph is simple. The definition of the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ for the graph \mathcal{G} is $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in E$, and $a_{ij} = 0$ otherwise.

Then, $L = [L_{ij}] \in \mathbb{R}^{N \times N}$ of graph \mathcal{G} is defined as $L_{ij} = \sum_{j=1}^N a_{ij}$ and $L_{ij} = -a_{ij}$. The degree matrix of graph \mathcal{G} with $d_i = \sum_{j=1}^N a_{ij}$ can also be expressed compactly as $L = D - A$, where $D = \text{diag}(d_1, \dots, d_N)$ is the degree matrix. An undirected graph also possesses a symmetric Laplacian matrix L with only real nonnegative eigenvalues.

C. Fractional Order System

Definition 1. The Caputo derivative of order $\alpha \in \mathbb{R}^+$ on the half axis \mathbb{R}^+ for function $f \in \mathbb{R}^{n \times n}$ is defined as follows:

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} = \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > 0 \quad (1)$$

In the [1], there are further details regarding fractional order systems.

Lemma 1[33]. Suppose that $V = \frac{1}{2}x^T P x$, $x \in \mathbb{R}^n$, $P > 0$, is a derivable and continuous function. Then, the following inequality is satisfied.

$${}_0^C D_t^\alpha V \leq x^T P {}_0^C D_t^\alpha x, \quad t \geq t_0 \quad (2)$$

Where $P \in \mathbb{R}^n$ is a constant, square, symmetric and positive definite matrix.

Lemma 2. If $f_i(x)$, $i = 1, \dots, k$ s.t $f_i(x) \in \mathbb{R}^n$ are convex functions, and $w_1, w_2, \dots, w_k \geq 0$, s.t $w \in \mathbb{R}^n$ then, the combination of convex functions such as $g(x)$ s.t $g(x) \in \mathbb{R}^n$ where $g(x) = w_1 f_1(x) + w_2 f_2(x) + \dots + w_k f_k(x)$ is convex function.

Proof: Based on the Jensen's inequality [34], while $k = 2$ then

$$\begin{aligned} g\left(\frac{x+y}{2}\right) &= w_1 f_1\left(\frac{x+y}{2}\right) + w_2 f_2\left(\frac{x+y}{2}\right) \\ &\leq w_1 \frac{f_1(x) + f_1(y)}{2} + w_2 \frac{f_2(x) + f_2(y)}{2} \\ &= \frac{g(x) + g(y)}{2} \end{aligned} \quad (3)$$

According to Lemma 2, if the stability of each topic of a convex system is proven, so the system by convex structure would be stable as well as [35]'s results. (q.e.d.)

Theorem 1[36]. consider the linear fractional order LTI system,

$${}_0^C D_t^\alpha x(t) = Ax(t), \quad x(0) = x_0 \quad (4)$$

Where $x(t) \in \mathbb{R}^n$ for all $t \geq 0$ and $A \in \mathbb{R}^{n \times n}$. The fractional order system LTI, where $0 < \alpha < 2$, is globally asymptotically stable if and only if

$$|\arg(\text{spec}(A))| > \frac{\alpha \pi}{2} \quad (5)$$

The corresponding condition of theorem can be expressed in terms of LMI when the fractional order α is within the interval $0 < \alpha < 1$. Further explanations on this topic can be found in reference [1].

III. MAIN PROBLEM

A multi-agent system with a leader-follower structure is investigated, composed of $N-1$ agents (referred to as the followers) indexed by $1, \dots, N-1$, and a single agent (referred to as the leader) indexed by N . The leader's dynamic is exemplified by

$$\begin{aligned} {}_0^C D_t^\alpha x_i(t) &= Ax_N(t), \\ y(t) &= C_1 x_N(t), \\ z_N(t) &= C_2 x_N(t) \end{aligned} \quad (6)$$

The dynamics of the followers are identical and are represented by

$$\begin{aligned} {}_0^C D_t^\alpha x_i(t) &= Ax_i(t) + Bu_i(t) + Sw_i(t), \\ y_i(t) &= C_1 x_i(t) + D_1 w_i(t), \\ z_i(t) &= C_2 x_i(t) + D_2 u_i(t), \\ i &= 1, \dots, N-1 \end{aligned} \quad (7)$$

In this study, we represent the state, measured output, output to be regulated, coupling input, and unknown external disturbance of the i^{th} follower using the variables $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^r$, $z_i \in \mathbb{R}^p$, $u_i \in$

\mathbb{R}^m , and $w_i \in \mathbb{R}^q$, in that specific order. The dimensions of the matrices A, B, C_1, C_2, D_1, D_2 , and S are conformant. We assume that the pair (C_1, A) is detectable, and the pair (A, B) is stabilizable. In this study, we investigate dynamic output feedback rule assuming that each follower has access to relative output data about its neighbors. We propose a distributed dynamic consensus rule based on observer feedback, specifically regarding [37].

$$\begin{aligned} {}_0^C D_t^\alpha \vartheta_i &= (A - GC_1)\vartheta_i + \sum_{j=1}^N a_{ij} [BF(\vartheta_i - \vartheta_j) + \\ &\quad G(y_i - y_j)] \\ &\quad \text{where} \\ &\quad u_i = F\vartheta_i, \\ &\quad i = 1, \dots, N-1, \end{aligned} \quad (8)$$

The state w_i is used to estimate the relative state given by the expression $\sum_{j=1}^N a_{ij}(x_i - x_j)$, where $G \in \mathbb{R}^{n \times r}$ and matrix and $F \in \mathbb{R}^{m \times n}$ is a $m \times n$ matrix used for local feedback control. It is important to note that $\vartheta_N = 0$ (the leader does not take any control signals), where a_{ij} is the ij^{th} entry of the adjacency matrix A associated with graph \mathcal{G} , and this satisfies the standard assumption. **Assumption 1:** No information is given to the leader by any followers. Any one of the $N-1$ followers can access the leader's state, and there is a linked, undirected, and simple communication graph among them. Given that the leader has no neighbors, graph \mathcal{G} 's Laplacian matrix can be partitioned as

$$L = \begin{bmatrix} L_1 & L_2 \\ 0_{1 \times (N-1)} & 0 \end{bmatrix} \quad (9)$$

where $L_2 \in \mathbb{R}^{(N-1) \times 1}$, and $L_1 \in \mathbb{R}^{(N-1) \times (N-1)}$.

Lemma 3[16]. Since L_1 is positive definite under Assumption 1, all of its eigenvalues have positive real components.

The consensus rule is mainly designed to deal with the issue of leader-follower consensus control for agents 1 and 2. We are concerned with the differences between the states of the leader and followers because it is important for followers to mimic the leader's state in the context of leader-follower consensus control. Therefore, we introduce a new error state variable for each follower, $e_i = x_i - x_N$, where consensus between the leaders and followers is achieved when $e_i = 0$, meaning x_i approaches x_N as t approaches infinity for all $i = 1, \dots, N-1$. On the other hand, we are interested in the variations in the output values of the leader and followers in the context of distributed H_2 Robust control for multi-agent systems. In reality, the boyd's book delves deep into the best approach to managing H_2 [34]. Thus, the output disagreement between the leader and followers is reflected in the performance output variable, which is defined as $\eta_i = z_i - z_N$, $i = 1, \dots, N-1$. Indicate $e = [e_1^T, \dots, e_{N-1}^T]^T$, $\eta = [\eta_1^T, \dots, \eta_{N-1}^T]^T$, $\xi = [y_1^T - y_N^T, \dots, y_{N-1}^T - y_N^T]^T$, $u = [u_1^T, \dots, u_{N-1}^T]^T$, $w = [w_1^T, \dots, w_{N-1}^T]^T$ and $\vartheta = [\vartheta_1^T, \dots, \vartheta_{N-1}^T]^T$. The error system's dynamics can then be expressed as

$$\begin{aligned} {}_0^C D_t^\alpha e &= [I_{N-1} \otimes A]e + [I_{N-1} \otimes B]u + [I_{N-1} \otimes S]w \\ \xi &= [I_{N-1} \otimes C_1]e + [I_{N-1} \otimes D_1]w \\ \eta &= [I_{N-1} \otimes C_2]e + [I_{N-1} \otimes D_2]u \end{aligned} \quad (10)$$

Thus, the procedure (6) can be expressed as

$$\begin{aligned} {}_0^C D_t^\alpha \vartheta &= [I_{N-1} \otimes (A - GC_1)]\vartheta + [L_1 \otimes BF]w + [L_1 \otimes G]\xi \\ u &= [I_{N-1} \otimes F]\vartheta \end{aligned} \quad (11)$$

The controlled error system is obtained by integrating the dynamic consensus rule (11) with the error system (10).

$$\begin{aligned} \begin{bmatrix} {}_0^C D_t^\alpha e \\ {}_0^C D_t^\alpha \vartheta \end{bmatrix} &= \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ L_1 \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + L_1 \otimes BF \end{bmatrix} \begin{bmatrix} e \\ \vartheta \end{bmatrix} + \\ &\quad \begin{bmatrix} I_{N-1} \otimes S \\ L_1 \otimes GD_1 \end{bmatrix} w, \\ \eta &= [I_{N-1} \otimes C_2 I_{N-1} \otimes D_2 F] \begin{bmatrix} e \\ \vartheta \end{bmatrix} \end{aligned} \quad (12)$$

Determine $A_0 = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes B \\ L_1 \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + L_1 \otimes BF \end{bmatrix}$, $C_0 = [I_{N-1} \otimes C_2 \quad I_{N-1} \otimes D_2 F]$, $S_0 = \begin{bmatrix} I_{N-1} \otimes S \\ L_1 \otimes GD_1 \end{bmatrix}$. After that, the controlled error system (12)'s impulse response matrix from the external disturbance w to the performance output η is equal to

$$T_{F,G} = C_0 e^{A_0 t} S_0 \quad (13)$$

As a result, the related cost functional H_2 is provided by

$$J(F, G) := \int_0^\infty \text{tr}[T_{F,G}^T(t) T_{F,G}(t)] dt \quad (14)$$

This defines the square of the L_2 -norm of the impulse response as the system's performance (12). Due to the communication limitations among the agents, the H_2 optimal leader-follower consensus control problem is a nonconvex optimization problem, and it is unknown at this time whether a closed-form solution exists in the literature. As an alternative, we find a solution to the problem that just needs suboptimality.

Definition2. The distributed H_2 suboptimal leader-follower consensus problem for the multi-agent system (6) and (7) is said to be solved by the consensus rule (8) if,

- In the event where all followers' external disturbances equal zero, or $w = 0$, we have $x_i \rightarrow x_N$ and $w_i \rightarrow 0$ for all $i = 1, \dots, N - 1$.
- $J(F, G) < \sigma$, where σ is a given upper bound.

Our aim is to address the following issue: First problem: Let $\sigma > 0$. In order to establish leader-follower consensus and $J(F, G) < \sigma$ in the dynamic consensus rule (8), design local feedback gain matrices $G \in \mathbb{R}^{n \times r}$ and $F \in \mathbb{R}^{m \times n}$.

IV. CONSENSUS PROTOCOL

We address first challenge in this part and present a design approach to derive gain matrices F and G . L_1 is a positive definite in accordance with Assumption1 and Lemma2, which suggests that L_1 is diagonalizable. Let $U \in \mathbb{R}^{(N-1) \times (N-1)}$ be an orthogonal matrix that diagonalizes the matrix L_1 , that is, $U^T L_1 U = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_{N-1})$, where the eigenvalues of L_1 are $\lambda_i > 0$, $i = 1, \dots, N - 1$. By means of the state transformation:

$$\begin{bmatrix} \hat{e} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} U^T \otimes I_n & 0 \\ 0 & U^T \otimes I_n \end{bmatrix} \begin{bmatrix} e \\ \eta \end{bmatrix} \quad (15)$$

the controlled error system (7) becomes

$$\begin{bmatrix} {}_0^C D_t^\alpha \hat{e} \\ {}_0^C D_t^\alpha \hat{\eta} \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes B \\ \Lambda \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + \Lambda \otimes BF \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} U^T \otimes S \\ U^T L_1 \otimes GD_1 \end{bmatrix} w, \eta = [U \otimes C_2 \quad U \otimes D_2 F] \begin{bmatrix} \hat{e} \\ \hat{\eta} \end{bmatrix} \quad (16)$$

It should be noted that the impulse response matrix from the disturbance input w to the output η remains equal to the impulse response matrix (13) following the transformation (15). In order to continue, the $N - 1$ auxiliary linear systems listed below are presented:

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{e}_i(t) &= A \tilde{e}_i(t) + B \tilde{u}_i(t) + S \tilde{w}_i(t) \\ {}_0^C D_t^\alpha \tilde{\xi}_i(t) &= C_1 \tilde{e}_i(t) + D_1 \tilde{w}_i(t) \quad i = 1, \dots, N - 1 \\ \tilde{\eta}_i(t) &= C_2 \tilde{e}_i(t) + D_2 \tilde{u}_i(t) \end{aligned} \quad (17)$$

where the state, coupling input, external disturbance, measured output, and output to be controlled of the i^{th} auxiliary system are, respectively, represented by the variables $\tilde{e}_i \in \mathbb{R}^n$, $\tilde{u}_i \in \mathbb{R}^m$, $\tilde{w}_i \in \mathbb{R}^q$, $\tilde{\xi}_i \in \mathbb{R}^r$, and $\tilde{\eta}_i \in \mathbb{R}^p$. with the use of the corresponding dynamic feedback controllers.

$${}_0^C D_t^\alpha \tilde{\theta}_i = A \tilde{\theta}_i + B \tilde{u}_i + G(\tilde{\xi}_i - C_1 \tilde{\theta}_i) \quad (18)$$

$i = 1, \dots, N - 1$
Since the eigenvalues of L_1 , $\lambda_i > 0$, $i = 1, \dots, N - 1$, are expressed, the closed-loop systems can be expressed as

$$\begin{bmatrix} {}_0^C D_t^\alpha \tilde{e}_i \\ {}_0^C D_t^\alpha \tilde{\theta}_i \end{bmatrix} = \begin{bmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{bmatrix} \begin{bmatrix} \tilde{e}_i \\ \tilde{\theta}_i \end{bmatrix} + \begin{bmatrix} S \\ GD_1 \end{bmatrix} w \quad (19)$$

$$\tilde{\eta}_i = [C_2 \quad \lambda_i D_2 F] \begin{bmatrix} \tilde{e}_i \\ \tilde{\theta}_i \end{bmatrix}, i = 1, \dots, N - 1$$

Indicate that $\bar{A}_i = \begin{bmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{bmatrix}$, $\bar{C}_i = [C_2 \quad \lambda_i D_2 F]$, $\bar{S}_i = \begin{bmatrix} S \\ GD_1 \end{bmatrix}$. $\tilde{T}_{i,F,G}(t) = \bar{C}_i e^{\bar{A}_i t} \bar{S}_i$ is the impulse response matrix for each system (19), from the disturbance \tilde{w}_i to the output $\tilde{\eta}_i$. The corresponding H_2 cost functional is provided by $J_i(F, G) := \int_0^\infty [\tilde{T}_{i,F,G}^T(t) \tilde{T}_{i,F,G}(t)] dt$, $i = 1, \dots, N - 1$. Consequently, the following theorem holds.

Theorem2. Consider the fractional order closed loop system (20), if there exists F, G for $i = 1, \dots, N - 1$ and $|\arg(\text{spec}(\bar{A}_i))| > \alpha\pi/2$, with $\bar{A}_i = \begin{bmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{bmatrix}$, by bounded disturbance $\|w(t)\| \leq W$, then the closed-loop system for any agent in multi-agent fractional order system is asymptotically stable.

Proof. Consider the closed loop fractional order control system (20),

$${}_0^C D_t^\alpha \tilde{e}_i = A \tilde{e}_i + \lambda_i BF \tilde{\theta}_i + S w \quad (20)$$

The time solution of (20) is expressible in terms of the Mittag-Leffler function $E_{\alpha,\beta}$ (with $E_{\alpha,\beta} = E_{\alpha,1}$)

$$\begin{aligned} \tilde{e}_i(t) &= E_{\alpha,\alpha}(At^\alpha) \tilde{e}_i(0) \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^\alpha) [\lambda_i BF \tilde{\theta}_i(\tau)] [S w(\tau)] d\tau \end{aligned} \quad (21)$$

By taking L_2 norm of both side of equation (22) by assist of Grownwall-Bellman lemma in [38] yields to

$$\begin{aligned} \|\tilde{e}_i(t)\| &\leq (\|E_{\alpha,\alpha}(At^\alpha)\| \|\tilde{e}_i(0)\| + \|\lambda_i BF\| \|S\| \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\tau)^\alpha)\| \cdot \|\tilde{\theta}_i(\tau)\| \|w(\tau)\| d\tau) \\ &\leq (\|E_{\alpha,\alpha}(At^\alpha)\| \|\tilde{e}_i(0)\| + \|\lambda_i BF\| \|S\| \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\tau)^\alpha)\| \|\tilde{\theta}_i(\tau)\| \|w(\tau)\| d\tau) \end{aligned} \quad (22)$$

In this situation, if A be a specified Hurwitz matrix, and $\|\lambda_i BF\| \|S\|$ is bounded, then there exists a positive constant M such that

$$\begin{aligned} \|\lambda_i BF\| \|S\| \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\tau)^\alpha)\| \\ \leq \|\lambda_i BF\| \|S\| \int_0^t Q e^{c\tau} d\tau \leq M \end{aligned} \quad (23)$$

With $t \geq 0$ and $c > 0$. Because there is a positive constant M that $\|\lambda_i BF\| \|S\| \int_0^t Q e^{c\tau} d\tau \leq M$ for all $t > 0$, and $\|E_{\alpha,\alpha}(At^\alpha)\| \|\tilde{e}_i(0)\| \leq Q e^{ct}$ tends to 0 while $t \rightarrow \infty$ then we can conclude that

$$\|\tilde{e}_i(t)\| \leq M \cdot \|W\| \int_0^t (\tau)^{\alpha-1} \|\tilde{\theta}_i(\tau)\| d(\tau) \leq N \quad (24)$$

$\lim_{t \rightarrow \infty} \tilde{e}_i(t) \leq N + Q e^{-ct} < \infty$, then $\tilde{e}_i(t)$ is bounded. According to Jensen's inequality and Lemma1, the combination of multi-agent convex system also is bounded because each vertex is bounded. (q.e.d.).

This allows us to discuss the boundedness of signal W and, of course, the convergence of $\tilde{\theta}_i$ in (19) or augmented signal of \tilde{e}_i , $\tilde{\theta}_i$,

which is not achievable because of the screen's constraints. In the following theorem, we will discuss the process of designing the controller.

Theorem3: Assume that $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times r}$. Suppose that $D_2^T D_2 = I_m$, $D_1 D_1^T = I_r$, and $D_1 S^T = 0$. When all $N - 1$ systems in (19), are internally stabilized by the controllers (18) with the identical F, G , then the dynamic consensus rule (8) with gain matrices F, G achieves leader-follower consensus for the agents (6) and (7), and we have $J(F, G) = \sum_{i=1}^{N-1} J_i(F, G)$ is another result. Proof. From (15), we may deduce that $\hat{e} = 0$ and $\hat{\theta} = 0$ if and only if $e_i = 0$, $\vartheta_i = 0$, that is, $x_i \rightarrow x_N$ and $\vartheta_i \rightarrow 0$ for all $i = 1, \dots, N - 1$. Consequently, the leader-follower consensus issue can only be resolved if and only if $\lim_{t \rightarrow \infty} \hat{e}(t) = 0$ and $\lim_{t \rightarrow \infty} \hat{\theta}(t) = 0$. Remember that $U^T L_1 U = \theta$, and employing these two transformations.

$$\hat{w} = (U^T \otimes I_n) w, \quad \hat{\eta} = (U^T \otimes I_n) \eta \quad (25)$$

Converting the controlled error system (16) into as

$$\begin{bmatrix} \dot{\hat{e}} \\ \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ \Lambda \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + \Lambda \otimes BF \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\theta} \end{bmatrix} + \begin{bmatrix} I_{N-1} \otimes S \\ \Lambda \otimes GD_1 \end{bmatrix} \hat{w}, \hat{\eta} = \begin{bmatrix} I_{N-1} \otimes C_2 & I_{N-1} \otimes D_2 F \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\theta} \end{bmatrix} \quad (26)$$

Denote $\hat{A}_0 = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ \Lambda \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + \Lambda \otimes BF \end{bmatrix}$, $\hat{C}_0 = \begin{bmatrix} I_{N-1} \otimes C_2 & I_{N-1} \otimes D_2 F \end{bmatrix}$, $\hat{S}_0 = \begin{bmatrix} I_{N-1} \otimes S \\ \Lambda \otimes GD_1 \end{bmatrix}$. It is clear that the demolished subsystems $(\hat{A}_0, \hat{S}_0, \hat{C}_0)$ in (26) and the auxiliary systems $(\tilde{A}_i, \tilde{S}_i, \tilde{C}_i)$ in (19) are isomorphic for $i = 1, \dots, N - 1$. Thus, if and only if $\tilde{e}_1 = \dots = \tilde{e}_{N-1} = 0$ and $\tilde{\vartheta}_1 = \dots = \tilde{\vartheta}_{N-1}$ and, then and $\lim_{t \rightarrow \infty} \hat{e}(t) = 0$ and $\lim_{t \rightarrow \infty} \hat{\theta}(t) = 0$ respectively. Consider that $\rho_i = \tilde{e}_i - \tilde{\vartheta}_i$, and by using the transformation $\begin{bmatrix} \tilde{\rho}_i \\ \tilde{\vartheta}_i \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \tilde{e}_i \\ \tilde{\vartheta}_i \end{bmatrix}$, then $\tilde{A}_i = \begin{bmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{bmatrix}$ in (19) will be transformed into $\tilde{A}_{\rho_i} = \begin{bmatrix} A + \lambda_i BF & -GC_1 \\ 0 & A - GC_1 \end{bmatrix}$. This makes it clear that the \tilde{e}_i and $\tilde{\vartheta}_i$ states for $i = 1, \dots, N - 1$ converge asymptotically to zero if and only if the $N - 1$ systems' matrices $A + \lambda_i BK$ and $A - GC_1$ are stable. The consensus between the leader and followers is then reached. Next, we demonstrate that $J(F, G) := \sum_{i=1}^{N-1} J_i(F, G)$. Let F, G be such that matrices $A + \lambda_i BF$, matrix $A - GC_1$ are Hurwitz. Note that $U^T C_0 U = \hat{C}_0 U^T S_0 U = \hat{S}_0$, $U^T A_0 U = \hat{A}_0$, where $U^T = \begin{bmatrix} U^T \otimes I_n & 0 \\ 0 & U^T \otimes I_n \end{bmatrix}$. Next, we get by changing (13) in (14),

$$\begin{aligned} J(F, G) &:= \int_0^\infty \text{tr} [T_{F,G}^T(t) T_{F,G}(t)] dt = \\ & \int_0^\infty \text{tr} [(C_0 e^{A_0 t} S_0)^T (C_0 e^{A_0 t} S_0)] dt \\ &= \int_0^\infty \text{tr} [U (\hat{C}_0 e^{\hat{A}_0 t} \hat{S}_0)^T (\hat{C}_0 e^{\hat{A}_0 t} \hat{S}_0) U^T] dt \end{aligned} \quad (27)$$

Remember that the decomposed subsystems $(\hat{A}_i, \hat{S}_i, \hat{C}_i)$ in (26) and auxiliary systems $(\tilde{A}_i, \tilde{S}_i, \tilde{C}_i)$ in (19) are isomorphic. Specifically, $U^T L_1 U = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_{N-1})$, $D_1 S^T = 0$, $D_2^T C_2 = 0$, $D_1 D_1^T = I_r$, $D_2^T D_2 = I_m$. Thus, for $i = 1, \dots, N - 1$,

$$1, \text{tr} (\hat{S}_{0i}^T e^{\hat{A}_{0i} t} \hat{C}_{0i}^T \hat{C}_{0i} e^{\hat{A}_{0i} t} \hat{S}_{0i}) = \text{tr} (\tilde{S}_i^T e^{\tilde{A}_i t} \tilde{C}_i^T \tilde{C}_i e^{\tilde{A}_i t} \tilde{S}_i).$$

Consequently

$$\begin{aligned} J(F, G) &= \int_0^\infty \sum_{i=0}^{N-1} \text{tr} (U \tilde{S}_i^T e^{\tilde{A}_i t} \tilde{C}_i^T \tilde{C}_i e^{\tilde{A}_i t} \tilde{S}_i U^T) dt = \\ & \int_0^\infty \sum_{i=1}^{N-1} \text{tr} (\tilde{T}_{i,F,G}(t)^T \tilde{T}_{i,F,G}(t)) dt = \sum_{i=1}^{N-1} J_i(F, G) \end{aligned} \quad (28)$$

The proof is now complete.

(q.e.d.).

The assumptions $D_1 D_1^T = I_r$ and $D_2^T D_2 = I_m$ in Theorem3 are merely for the purpose of simplifying the notation; they can be readily relaxed to the regularity conditions $D_1 D_1^T > 0$ and $D_2^T D_2 > 0$. A dynamic output feedback controller (18) can be used to recast the distributed H_2 suboptimal leader-follower consensus problem for the multi-agent system (6) and (7) into H_2 suboptimal control problems of $N - 1$ independent systems (19). This is done by applying Theorem3. Next, we demonstrate that the following lemma is given to address the problem of H_2 suboptimal control for $N - 1$ systems (19), i.e., all $N - 1$ systems are internally stable, while $\sum_{i=1}^{N-1} J_i(F, G) < \sigma$, for specified gain matrices $G \in \mathbb{R}^{n \times r}$ and $F \in \mathbb{R}^{m \times n}$.

Lemma4. If and only if $P_i > 0$, $i = 1, \dots, N - 1$, and $Q > 0$ fulfill, then the dynamic controllers (12) internally stabilize all $N - 1$ systems (19) and $\sum_{i=1}^{N-1} J_i(F, G) < \sigma$.

$$(A + \lambda_i BF)^T P_i + P_i (A + \lambda_i BF) + (C_2 + \lambda_i D_2 F)^T (C_2 + \lambda_i D_2 F) < 0 \quad (29)$$

$$QA + QA^T - QC_1^T C_1 Q + SS^T < 0 \quad (30)$$

$$\sum_{i=1}^{N-1} [\text{tr}(C_1 Q P_i Q C_1^T) + \text{tr}(C_2 Q C_2^T)] < \sigma \quad (31)$$

Proof (Sketch). The proof derives from the proof of lemma2 in [39], which is omitted here, but based on the following substitutions: $\bar{A} = A$, $\bar{B} = \lambda_i B$, $\bar{C}_1 = C_1$, $\bar{D}_1 = D_1$, $\bar{C}_2 = C_2$, $\bar{D}_2 = \lambda_i D_2$, and $\bar{S} = S$. However, take note that Lemma4 does not yet offer a way to calculate the matrices F, G . Then, the subsequent theorem offers a design technique for identifying appropriate matrices F, G . **Theorem4:** Let $D_1 S = 0$, $D_2^T C_2 = 0$, $D_1 D_1^T = I_r$, and $D_2^T D_2 = I_m$ be assumed. Assume $\sigma > 0$. Think about the linked H_2 cost functional (14), which is part of the controlled error system (10). Assume $Q > 0$ fulfills

$$AQ + QAT - QC_1^T C_1 Q + SS^T < 0. \quad (32)$$

Furthermore, consider the following two cases:

- 1) If $0 < c < \frac{2}{\lambda_1 + \lambda_{N-1}}$, where λ_{N-1} represents L_1 's greatest eigenvalue and λ_1 its smallest. After that, $P > 0$ is such that $A^T P + PA + (c^2 \lambda_1^2 - 2c \lambda_1) P B B^T P + C_2^T C_2 < 0$
- 2) There exists $P > 0$ satisfying $A^T P + PA + (c^2 \lambda_{N-1}^2 - 2c \lambda_{N-1}) P B B^T P + C_2^T C_2 < 0$ if $\frac{2}{\lambda_1 + \lambda_{N-1}} \leq c < \frac{2}{\lambda_{N-1}}$.

In both cases, if P and Q also satisfy in both scenarios, if P and Q further satisfy

$$\text{tr}(C_1 Q P Q C_1^T) + \text{tr}(C_2 Q C_2^T) < \frac{\sigma}{N-1} \quad (33)$$

For agents (6) and (7), leader-follower consensus is achieved by consensus rule (8) with $F = -c B^T P$ and $G = Q C_1^T$. However, the consensus rule is not ideal, as $J(F, G) < \sigma$. Proof. Note first that (31) is equal to (30). It is possible to verify that $c^2 \lambda_{N-1}^2 - 2c \lambda_{N-1} < 0$ for case 2) above by applying the upper and lower bound on c . The positive definite solution P exists for the Riccati inequality case 2). When $i = 1, \dots, N - 1$, $P_i = P$ and $F = -c B^T P$ are used in (29), the result is immediately

$$(A - c \lambda_i B B^T P)^T P + P (A - c \lambda_i B B^T P) + (C_2 - \lambda_i D_2 B^T P)^T (C_2 - \lambda_i D_2 B^T P) < 0 \quad (34)$$

Remember the circumstances This provides $(A - c \lambda_i B B^T P)^T P + P (A - c \lambda_i B B^T P) + c^2 \lambda_i^2 P B B^T P + C_2^T C_2 < 0$ since $D_2^T C_2 = 0$ and $D_2^T D_2 = I_m$. Given that $c^2 \lambda_1^2 - 2c \lambda_1 \leq c^2 \lambda_1^2 - 2c \lambda_1 \leq c^2 \lambda_{N-1}^2 - 2c \lambda_{N-1} < 0$ and $\lambda_i \leq \lambda_{N-1}$, for $i = 1, \dots, N - 1$, the positive definite solution P of case 2) also fulfills the $N - 1$ Riccati inequality.

$$A^T P + PA + (c^2 \lambda_1^2 - 2c \lambda_1) P B B^T P + C_2^T C_2 < 0 \quad (35)$$

subsequently it is evident from (33) that (31) is likewise true. $\sum_{i=1}^{N-1} J_i(F, G) < \sigma$ and all $N-1$ systems (19) are internally stabilized by Lemma 3. The consensus rule (8) thus attains leader-follower consensus for the agents (6) and (7) while $J(F, G) < \gamma$, as deduced from Theorem 2. The proof is identical and left out for case 1) above. (q.e.d.)

Remark 1. The distributed dynamic output consensus rule with gain matrices $F = -cB^T P$ and $G = QC_1^T$ is suboptimal, according to Theorem 4, provided that appropriate values of c , P , and Q are selected. Therefore, for this suboptimal problem, how may the upper bound σ be chosen as minimal as feasible so that $\text{tr}(C_1 Q P Q C_1^T) + \text{tr}(C_2 Q C_2^T) < \frac{\sigma}{N-1}$. It is simple to make the point that, generally speaking, smaller P and Q result in smaller $\text{tr}(C_1 Q P Q C_1^T) + \text{tr}(C_2 Q C_2^T)$, and subsequently, the smaller feasible given σ . One way to discover a tiny feasible σ would be to identify the smallest values of P and Q . When $\mu > 0$, we may prove that (31) is equal.

$$AQ + QA^T - QC_1^T C_1 Q + SS^T + \mu I_N = 0 \quad (36)$$

By using the standard argument, it can be shown that Q decreases as μ decreases. Consequently, if we chose $\mu > 0$ very close to 0, we can find a small solution for $Q(\mu) > 0$. Similarly, a small solution $P(c, \delta) > 0$ with $\delta > 0$ for the two cases 1) and 2) can be founded by establishing two equalities as follows

$$A^T P + PA - r_1 P B B^T P + C_2^T C_2 + \delta I_n = 0 \quad (37)$$

with

$$A^T P + PA - r_2 P B B^T P + C_2^T C_2 + \delta I_n = 0 \quad (38)$$

where $r_2 = (-c^2 \lambda_{N-1}^2 + 2c \lambda_{N-1})$ and $r_1 = (-c^2 \lambda_1^2 + 2c \lambda_1)$ are found. It follows that the lower P is, the greater r_1 (or r_2) and the smaller δ . It is possible to calculate that the maximum of r_1 occurs at $c^* = \frac{1}{\lambda_{N-1}}$, while the maximum of r_2 occurs at $c = \frac{1}{\lambda_1}$. Thus, in each of the two scenarios, we get the best solution to the Riccati inequalities case 1) and 2) as previously stated if we select $\delta > 0$ extremely near to 0 and $c = \frac{2}{\lambda_1 + \lambda_{N-1}}$. In the following, we will discuss the simulation and results.

V. SIMULATION AND RESULT

This section provides a simulation example to validate the effectiveness of our suggested consensus rule using dynamic output feedback. It examines a multi-agent leader-follower system with a leader and six followers as shown in Fig. 1,

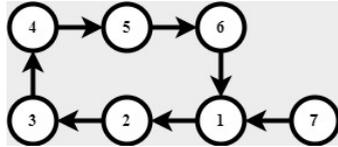


Fig. 1. The diagram of information between followers and leaders.

Where $A = \begin{bmatrix} -4 & 4 \\ -3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$, $S = \begin{bmatrix} 0.6 & 0 \\ 1 & 0 \end{bmatrix}$, $C_1 = [1 \ 1]$, $C_2 = \begin{bmatrix} 1 & 1.2 \\ 1 & 1 \end{bmatrix}$, $D_1 = [0 \ 1]$, $D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and also $D_1 S^T = [1 \ 1]^T$, $D_2^T C_2 = [1 \ 1]$, $D_1 D_1^T = 1$, $D_2^T D_2 = 1$. Let's use the graph \mathcal{G} shown in Fig. 1, in which node 7, As a result of the particular partition form of the Laplacian matrix L in (9) connected to graph \mathcal{G} , the matrix L_1 's eigenvalues are, $\lambda_{L_1} = \{0.1088, 1.0, 1.2954, 3.0, 3.3174, 4.2784\}$. We solved the H_2 leader-follower consensus control problem by computing the gain matrices F , G of the dynamic output feedback rule (8) where $F = [0.5415 \ -1.8970]$ and $G = \begin{bmatrix} 1.4559 \\ 1.6973 \end{bmatrix}$ using the technique suggested in Theorem 4. Let $\sigma = 100$ be the intended upper bound for the H_2 cost (13) in this case. We first compute a solution $P > 0$

in instance case 2) by solving the following using Theorem 4 while we reached to 13.77 as a norm of closed loop multi-agent fractional order control systems. In the following consensus control convergence of leader and follower depicted in Fig 2.

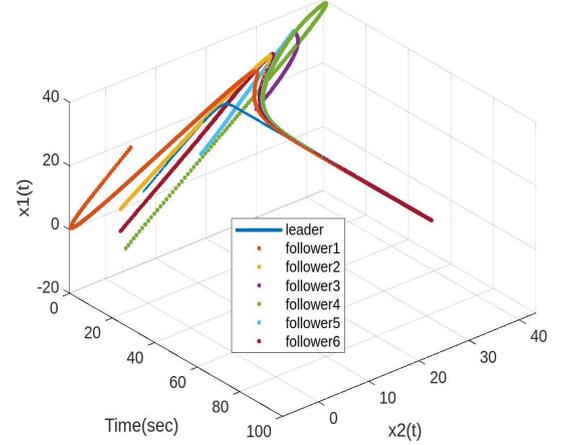


Fig 2. Convergence of followers to the leader through a consensus fractional-order control rule (12)

Certainly, the control system has achieved appropriate convergence, indicating acceptable performance with minimal energy consumption as shown in Fig 3.

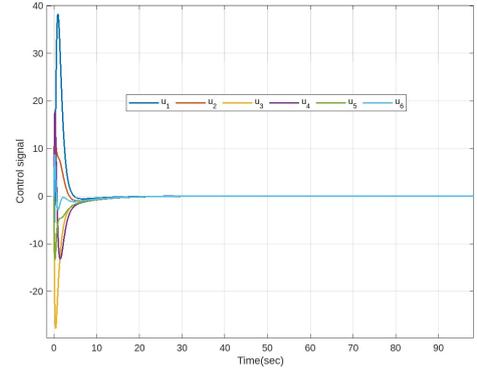
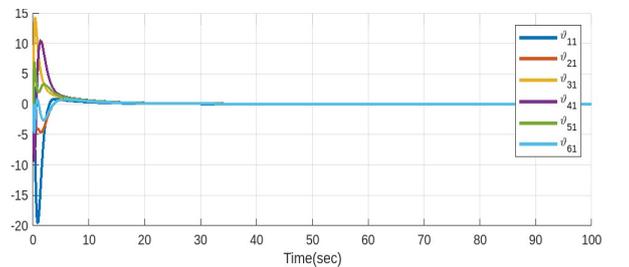


Fig4. The performance of the designed control signal of dynamic output feedback

The control system design has demonstrated suitable convergence and also states' convergence of the first and second states of the agents ϑ following the leader is illustrated in Fig 4. This fig depicted that the control system exhibits acceptable performance



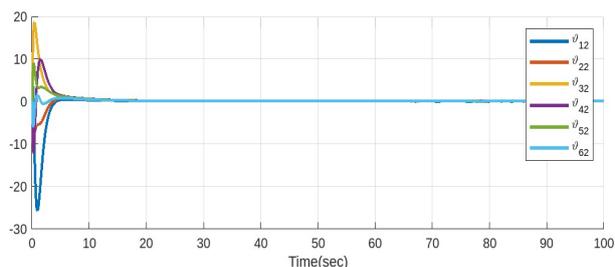


Fig4. the state vector convergence of dynamic output rule in convergence of follower to leader

VI. CONCLUSION

In this study, we used fractional order dynamic output feedback control to investigate the distributed H_2 suboptimal leader-follower consensus control problem. We focused on a multi-agent system with N agents, each consisting of an autonomous leader and $N - 1$ followers. Our approach involves designing a distributed protocol that achieves H_2 suboptimal leader-follower consensus. This means that the connected H_2 cost is lower than the specified upper bound, and the states of the followers converge to the leader's state.

ACKNOWLEDGMENT

This work has been supported by the LIT Secure and Correct Systems Lab funded by the State of Upper Austria and the Linz Institute of Technology (LIT-2019-7-INC-316).

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