Distributed Observer-Based Control for Consensus in Nonlinear Fractional-Order Multi-Agent Systems

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***Abstract*-The consensus problem among agents has long intrigued researchers in the field of coordination control. This study investigates a nonlinear fractional-order system with $0 < \alpha < 1$. A controller based on distributed observers is designed to assist agents in achieving consensus within a multi-agent nonlinear fractional-order system. The primary strategy for addressing the nonlinear term involves feedback linearization. The controller is developed with a stability proof, and the resulting observer-based controller is applied to an example of a nonlinear fractional-order multi-agent system. Additionally, a nonlinear example is solved analytically in this context. The estimation error is thoroughly analyzed, demonstrating notable convergence to zero based on the Lyapunov stability synthesis.**

Keywords: feedback linearization, nonlinear fractional order system, distributed mechanism, observer-based control, multi-agent system.

1 INTRODUCTION

In recent decades, the control of multi-agent systems has attracted considerable attention from researchers in both theoretical and industrial contexts. Many real-world systems, including unmanned aerial vehicles, multi-robot coordination (Yoo & Kim, 2017), and distributed wireless sensor networks (W. Li et al., 2017), can be considered as multi-agent systems. Consensus is a fundamental issue for multi-agent systems, referring to the scenario where all agents reach common states based on their local information. Consensus is divided into two forms: leaderless group consensus and leader-follower consensus (Wu et al., 2017) ,(Wang et al., 2017). There have been substantial efforts to achieve consensus in multi-agent systems, considering different types of agent dynamics (Ren et al., 2016), interaction topology (Zhang & Wang, 2017), communication environment (Meng et al., 2018), and other factors (Fang et al., 2016). When implementing a consensus control law, it is crucial to account for energy resource limitations, as the restricted computational speed and communication network bandwidth pose challenges in balancing control performance with energy efficiency. Periodic measurement and control based on event changes can be effective in conserving resources. In (Åström & Bernhardsson, 1999), a comparison was made between a controller using periodic sampling and one employing event-based control for first-order systems. The study demonstrated that event-based control outperforms the time-stimulation strategy in terms of reducing communication costs. (Dimarogonas & Johansson, 2009) applied the event triggering method in multi-agent systems for the first time. In sampled-data control, the controller for each agent is updated at discrete time intervals with both periodic and non-periodic sampling methods widely adopted in numerous studies (Chen & Li, 2014), (Fan et al., 2016). By employing a distributed strategy based on event stimulation, the controller is updated only when a predefined event occurs (Duan et al., 2017), (Wu et al., 2017). Achieving event-based consensus in multi-agent systems with nonlinear dynamics continues to be a challenging problem. There have been few recent studies on this issue. One study by (Wu et al., 2017), investigates distributed event-based control with both fixed and switching topologies. It introduces a new event-triggering rule to reduce the communication load in multi-agent networks. Another study by (Liuzza et al., 2013) considers a relatively general class of multi-agent systems with nonlinear dynamics and proposes a model-based approach. The consensus problem of multi-agent leader-follower systems with nonlinear agents and an event-triggered control mechanism is discussed in (Su et al., 2017), (Rehák & Lynnyk, 2021a). The method proposed in this paper achieves more favorable results, as indicated by the system simulation presented in (Su et al., 2017), which shows the number of control signal updates required to achieve consensus. The field of distributed control faces a significant challenge when dealing with multi-agent systems that contain nonlinear components. This manuscript focuses on a specific type of fractional-order multi-agent system with nonlinear trem and discusses the use of the feedback linearization method to address this issue. The paper begins by introducing concepts such as multi-agent systems, fractional-order systems, and then proceeds to examine the problem in the form of a theorem.

2 PRELIMINARIES

2.1 Notation:

The symbols $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$ represent the field of real numbers, n-dimensional real vector space, and $m \times n$ real matrices, respectively. The superscript $\intercal$ denotes transposition for matrices and vectors. The $n \times n$ identity matrix is written as I. The trace of a square matrix A is represented as $\mathrm{tr}(A)$. If all eigenvalues of a matrix have negative real parts, the matrix is called Hurwitz (or stable). For a symmetric matrix P, we write $P < 0$ for negative definiteness and $P > 0$ for positive definiteness. A diagonal matrix of size $n \times n$ with entries $(d_1, \dots, d_n)$ on the diagonal is represented by $\mathrm{diag}(d_1, \dots, d_n)$. Let blockdiag $(M_1, \dots, M_m)$ be the block diagonal matrix with diagonal blocks M_i for matrices $M_1, \dots, M_m$. $A \otimes B$ represents the Kronecker product of matrix A and matrix B.

2.2 Fractional Order System:

Definition 1. The Caputo derivative of order $\alpha \in \mathbb{R}^+$ on the half axis $\mathbb{R}^+$for function $f \in \mathbb{R}^{n\times n}$ is defined as follows:

$$ {}_{0}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\quad t>0 \qquad (1) $$

We refer to (Petráš, 2006) and chapter 6 of (Atangana, 2017) for further details regarding fractional order systems, beyond what is necessary for our exposition and the reviewed hereafter.

Definition 2 (Definition 8 in Bakkyaraj, 2020). The partial Caputo derivative of a function $f\in\mathbb{R}^{n\times n}$ order $\alpha\in\mathbb{R}^{+}$ on the half axis $\mathbb{R}^{+}$ is defined as follows:

$$ {}_{0}^{C}\partial_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{\left(\frac{\partial}{\partial\tau}\right)^{(n)}f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, t>0 \qquad (2) $$

Lemma 1(Duarte-Mermoud et al., 2015). Consider a continuously differentiable function $V=\frac{1}{2}x^{T}Px$, where $x\in\mathbb{R}^{n}$, and P is a positive definite matrix $P>0$. Such a function will satisfy the following inequality:

$$ {}_{0}^{C}D_{t}^{\alpha}V\leq x^{T}P\,{}_{0}^{C}D_{t}^{\alpha}x,\quad t\geq t_{0} \qquad (3) $$

Lemma 2. Let $f_i(x), i=1,\dots,k$, be convex functions, where $f_i(x)\in\mathbb{R}^{n}$ for $x\in\mathbb{R}^{n}$. Suppose $w_i\geq 0$ for all $i=1,..,k$, consider $g(x)\in\mathbb{R}^{n}$ as $g(x)=w_1f_1(x)+w_2f_2(x)+\dots+w_kf_k(x)$ where $w_i\in\mathbb{R}^{n}$. Then $g(x)$ is a convex function.

Proof: Based on the Jensen's inequality (Boyd & Vandenberghe, 2010), while $k=2$ then

$$ g\left(\frac{x+y}{2}\right)=w_1f_1\left(\frac{x+y}{2}\right)+w_2f_2\left(\frac{x+y}{2}\right)\leq w_1\frac{f_1(x)+f_1(y)}{2}+w_2\frac{f_2(x)+f_2(y)}{2}=\frac{g(x)+g(y)}{2} \qquad (4) $$

If the stability of each topic in a convex system is stable, the system with a convex structure and then outcomes of is stable, the entire description is provided in (Apkarian et al., 1995).

Theorem 1(Lim et al., 2012). Consider the linear fractional order system,

$$ {}_{0}^{C}D_{t}^{\alpha}x(t)=Ax(t),\quad x(0)=x_0 \qquad (5) $$

Where $A\in\mathbb{R}^{n\times n}$ and $x(t)\in\mathbb{R}^{n}$ for all $t\geq 0$. When $0<\alpha<2$, the fractional order system LTI is globally asymptotically stable if and only if

$$ |\arg(\mathrm{spec}(A))|>\frac{\alpha\,\pi}{2} \qquad (6) $$

The corresponding condition of the Theorem 1 can be expressed in terms of a LMI when the fractional order α is within the interval $0<\alpha<1$. Further explanations on this topic can be found in reference (Petráš, 2006) and (Shamaghdari, 2024).

3 MAIN PROBLEM

Consider two smooth functions, f, g, such that $f(0)=0$ and $g(0)\neq 0$. Given two Jacobian matrices, A and C, evaluate to the origin and h, respectively.

Assumption 1. The pair (C, A) is observable.

The multi-agent system is structured by N distinct agents like:

$$ \begin{aligned} {}_{0}^{C}D_{t}^{\alpha}x_i&=f(x_i)+g(x_i)u_i\\ y_i&=h(x_i) \end{aligned} \qquad (7) $$

The objective is to design a consensus control law u_i for each agnt such that the following holds with $\bar{x}=\frac{1}{N}\sum_{i=1}^{N}x_i$,.

$$ \lim_{t\to\infty}\|x_i(t)-\bar{x}(t)\|=0 \qquad (8) $$

The output y is considered to be the only quantifiable variable. Additionally, an observer is required to estimate the state, as the controller needs to know it. This means that the control law of the *i*th agent is:

$$ u_i=\mathcal{K}\sum_{j=1}^{N}\left(T(\hat{x}_j)-T(\hat{x}_i)\right), \qquad (9) $$

Where, $\hat{x}_i$ is the state estimate of the i^{th} agent provided by a state observer (to be proposed), $T:\mathbb{R}^{n}\to\mathbb{R}^{n}$ is a diffeomorphism, which is a differentiable homeomorphism—i.e., a function that is continuously differentiable and has a continuously differentiable inverse (Khalil, 2002). The control gain, denoted as $\mathcal{K}$, will be computed in the subsequent section. For each agent, the state observer is defined as follows:

$$ {}_{0}^{C}D_{t}^{\alpha}\hat{x}_i=f(\hat{x}_i)+g(\hat{x}_i)u_i+\mathcal{L}(\hat{x}_i)(y-h(\hat{x}_i)) \qquad (10) $$

The following section provides an idea for determining the observer gain $\mathcal{L}$.

3.1.1 NONLINEAR OBSERVER

The nonlinear Luenberger observer was initially introduced by (Kazantzis & Kravaris, 1998) and promoted observer is inspired by this paper. Matrix $\bar{A}\in\mathbb{R}^{n\times n}$ is initially selected in such that

$$ \max\mathrm{Re}\ \mathrm{eig}(\bar{A})<\min\left(\min\mathrm{Re}\left(\mathrm{eig}\left(\frac{{}_{0}^{C}\partial_{t}^{\alpha}f}{{}_{0}^{C}\partial_{t}^{\alpha}x}(0)\right)\right),0\right) \qquad (11) $$

Additionally, by selecting a vector b such that the pair $(\bar{A}, b)$ is controllable, assign each agent a function $\mu:\mathbb{R}^{n}\to\mathbb{R}^{n}$ that satisfies the equation:

$$ \frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}f(x)=\bar{A}x+bh(x),\qquad \mu(0)=0, \qquad (12) $$

The observer gain is thus expressed as follows for $x^{*}\in\mathbb{R}^{n}$:

$$ \mathcal{L}(x^{*})=\left(\frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}(x^{*})\right)^{-1}b \qquad (13) $$

Given the definition, let the observer of the i^{th} agent be expressed as:

$$ \hat{x}_i=f(\hat{x}_i)+g(\hat{x}_i)u_i+\mathcal{L}(\hat{x}_i)\big(h(x_i)-h(\hat{x}_i)\big) \qquad (14) $$

(Kazantzis & Kravaris, 1998) show that, if $u=0$, then $\lim_{t\to\infty}\|x(t)-\hat{x}(t)\|=0$, even without a control, provided the function $g(\hat{x})$ is constant. It is worth noting that function μ has a non-singular Jacobian matrix in a neighbourhood around the origin. As a result, the observer gain is appropriately defined in that region. Furthermore, this observation suggests that the mapping μ is a diffeomorphism. Further in-depth study is required if this criterion is not met. The method is based on the evidence presented by (Kazantzis & Kravaris, 1998). For example, $z_i=\mu(x_i)$, $\hat{z}_i=\mu(\hat{x}_i)$. so this means that

$$ \begin{aligned} &{}_{0}^{C}D_{t}^{\alpha}z_i-{}_{0}^{C}D_{t}^{\alpha}\hat{z}_i=\\ &\frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}(x_i)(f(x_i)+g(x_i)u_i)-\\ &\frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}(\hat{x}_i)\left(f(\hat{x}_i)+g(\hat{x}_i)u+\mathcal{L}(\hat{x}_i)\big(h(x_i)-h(\hat{x}_i)\big)\right) \qquad (15)\\ &=\bar{A}\big(\mu(x_i)-\mu(\hat{x}_i)\big)\\ &+\left(\frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}(x_i)g(x_i)-\frac{{}_{0}^{C}\partial_{t}^{\alpha}\mu}{{}_{0}^{C}\partial_{t}^{\alpha}x}(\hat{x}_i)g(\hat{x}_i)\right)u_i \end{aligned} $$

Assumption 2. For each $x^*, x^{**} \in \mathbb{R}^n$, there is a positive constant $M > 0$

$$\left\| \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(x^*)g(x^*) - \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(x^{**})g(x^{**}) \right\| \leq M\|x^* - x^{**}\| \quad (16)$$

Relation (16) and the fact that function μ is a diffeomorphism suggest that a constant $M^* > 0$ exists, meaning that

$$\left\| \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(x_i)g(x_i) - \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(\hat{x}_i)g(\hat{x}_i) \right\| \leq M^*\|z_i - \hat{z}_i\| \quad (17)$$

Furthermore, matrix $\bar{A}$ selected as a Hurwitz, there exists a matrix $P \in \mathbb{R}^{n\times n}$, with $P > 0$, that satisfies the condition for all $\beta > 0$.

$$\bar{A}^T P + P\bar{A} = -\beta I_n \quad (18)$$

Define for $i = 1, \ldots, n$ Lyapunov functions $V_i = (z_i - \hat{z}_i)^T P(z_i - \hat{z}_i)$. Then

$${}_0^C D_t^\alpha V_i \leq -\beta(z_i - \hat{z}_i)^2 + M^*(z_i - \hat{z}_i)^2\|u_i\| \quad (19)$$

Therefore, for all $t \geq 0$, ${}_0^C D_t^\alpha V_i < 0$ if $\beta > M^*\|u_i\|$. (Kazantzis & Kravaris, 1998) demonstrated the existence of a solution to equation (12) under certain restrictive assumptions, which required the eigenvalues of the Jacobian matrix of f to lie in the left or right complex half-plane. Other proofs have introduced the requirements outlined in equation (11), and a finite element method proof was presented in (Rehák & Lynnyk, 2019) under less stringent assumptions.

3.1.2 CONSENSUS CONTROL FOR NONLINEAR FRACTIONAL ORDER MULTIAGENT SYSTEM

The precise feedback linearization is provided to each agent. In the following section, we address this issue by presenting our investigation of the problem as Theorem 2.

Assumption 3. System (7) has a relative degree of n. so consider that

$$\begin{cases} \theta_{1,i} = h(x_i) \\ \theta_{2,i} = (L_f h)(x_i) \\ \vdots \\ \theta_{n,i} = (L_f^{n-1} h)(x_i) \end{cases} \quad (20)$$

See (Khalil, 2002) for more information on Lie's derivative and its use feedback linearization. Additionally, let $\theta_i = \left(\theta_{i,1}, \ldots, \theta_{i,n}\right)^T$. Then, a pair of functions $\mu, \eta: \mathbb{R}^n \to \mathbb{R}$ exists such that

$${}_0^C D_t^\alpha \theta_{n,i} = \mu(\theta_i) + \eta(\theta)u_i\,. \quad (21)$$

consider that $\bar{\mu}_i = \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha \theta_i}(0) \in \mathbb{R}^{1\times n}$, $\tilde{\mu}(\theta_i) = \mu(\theta_i) - \bar{\mu}\theta_i$ and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$ with A being the coupling topology (adjacency matrix), be introduced as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = [\bar{\mu}_i]_{i=1}^n, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (22)$$

Furthermore, define the control transformation as $v_i = \tilde{\mu}(\theta_i) + \eta(\theta_i)u_i$. Next

$${}_0^C D_t^\alpha \theta_i = A\theta_i + B\nu_i \quad (23)$$

Let's $\nu = (\nu_1, \ldots, \nu_N)^T$ and $\theta = (\theta_1^T, \ldots, \theta_N^T)^T$. Additionally, define the mapping $T: \mathbb{R}^n \to \mathbb{R}^n$ where

$$T(x^*) = \left(h(x^*), (L_f h)(x^*), \ldots, (L_f^{n-1}h)(x^*)\right)^T.$$

Assumption 3 ensures that this arrangement is well-defined and establishes a diffeomorphism in a region near the origin. The goal is to achieve consensus in the set of systems given by (23). This is not particularly challenging because the systems being controlled are linear. Consider $\nu = (I \otimes \mathcal{K})\theta$ as the consensus control. Then, the system under control complies with the equation

$${}_0^C D_t^\alpha \theta = (I_N \otimes A)\theta + (\mathcal{L} \otimes B\mathcal{K})\theta \quad (24)$$

This translates into the (x, u) coordinates as

$$u_i = \frac{1}{\eta(T(x_i))}\left(\sum_{j=1}^N \mathcal{K}\left(T(x_j) - T(x_i)\right) - \tilde{\mu}(T(x_i))\right) \quad (25)$$

Due to the need for state awareness, this control is not achievable, even when coupling the multi-agent system described in equation (7). It is assumed that the i^{th} agent has access to the state estimates of its neighbors. As a result, the states x_i and x_j are replaced by the estimates provided by the observers. Consequently, equation (25) is substituted with:

$$u_i = \frac{1}{\eta(T(\hat{x}_i))}\left(\sum_{j=1}^N \mathcal{K}\left(T(\hat{x}_j) - T(x_i)\right) - \tilde{\mu}(T(\hat{x}_i))\right) \quad (26)$$

The current objective is to find the control gain $\mathcal{K}$. Using the converted system (23), we define the disagreement vector.

$$\delta = \theta - 1 \otimes \bar{\theta} \quad (27)$$

Given that the controller has access to the state θ, we can proceed with implementing state feedback. With state feedback, the disagreement dynamics follow the following formula:

$${}_0^C D_t^\alpha \delta = (I_N \otimes A)\delta + (\mathcal{L} + B\mathcal{K})\delta. \quad (28)$$

Using the well-known procedure (Z. Li et al., 2009), the control gain $\mathcal{K}$ can be designed such that $\lim_{t\to\infty}\|\delta\| = 0$. Therefore there exists a matrix $Y \in R^{n\times n}, Y > 0$ such that there exists a constant $\beta_w > 0$ satisfying:

$$A^T Y + YA = -\beta_W I_N \quad (29)$$

Hence, the function $W = \delta^T(I_N \otimes Y)\delta$ holds

$${}_0^C D_t^\alpha W \leq -\beta_w\|\delta\|^2 \quad (30)$$

The controller, however, does not have access to the state $x_i(t)$. Therefore, the vector θ should be replaced with an estimate $\hat{\theta} = T(\hat{x})$ as the vectors θ_i and θ are unavailable. Consequently, equation (24) is replaced with

$${}_0^C D_t^\alpha \theta = (I_N \otimes A)\theta + (\mathcal{L} + B\mathcal{K})\hat{\theta} \quad (31)$$

This suggests that (28) is swapped out for

$${}_0^C D_t^\alpha \delta = (I_N \otimes A)\delta + (\mathcal{L} \otimes B\mathcal{K})\delta + (\mathcal{L} \otimes B\mathcal{K})\left(\hat{\theta} - \theta\right) \quad (32)$$

This has led to the use of equation (30)

$${}_0^C D_t^\alpha W \leq -\beta_w\|\delta\|^2 + \delta^T(\mathcal{L} \otimes YB\mathcal{K})\left(\hat{\theta} - \theta\right) \quad (33)$$

The Lyapunov function $\mathcal{V}(\theta)$ can now be defined as:

$$\bar{V} = \sum_{i=1}^N V(z_i - \hat{z}_i) + W(\delta) \quad (34)$$

Afterward, its derivative is well-behaved.

$${}_0^C D_t^\alpha \bar{V} \leq \sum_{i=1}^n {}_0^C D_t^\alpha \bar{V}(z_i - \hat{z}_i) + {}_0^C D_t^\alpha W(\delta) \leq \sum_{i=1}^N -\beta(z_i - \hat{z}_i) + M^*(z_i - \hat{z}_i)^2\|u_i\| - \beta_w\|\delta\|^2 + \delta^T(\mathcal{L} \otimes YB\mathcal{K})\left(\hat{\theta} - \theta\right) \quad (35)$$

First, note that, since $\theta = T(x)$, $z_i = \mu(x_i)$, and both mappings μ and T are diffeomorphisms, there exists a constant $\varkappa > 0$ such that, with $z = [z_1^T, \dots, z_N^T]^T$ and $\hat{z} = [\hat{z}_1^T, \dots, \hat{z}_N^T]^T$, where

$$\varkappa\|\theta - \hat{\theta}\| \geq \|z - \hat{z}\| \tag{36}$$

Then

$$ {}_0^C D_t^\alpha \bar{V} \leq \sum_{i=1}^N -\beta\varkappa^2\|\theta - \hat{\theta}\|^2 + M^*\varkappa^2\|\theta - \hat{\theta}\|^2\|u_i\| - \beta_w\|\delta\|^2 + \delta^T(\mathcal{L} \otimes YB\mathcal{K})(\hat{\theta} - \theta). \tag{37}$$

Furthermore, a constant $\bar{\beta} > 0$ exists, meaning that

$$\delta^T(\mathcal{L} \otimes YB\mathcal{K})(\hat{\theta} - \theta) \leq \bar{\beta}\|\delta\|\|\theta - \hat{\theta}\|. \tag{38}$$

For each $\alpha > 0$,

$$\delta^T(\mathcal{L} \otimes YB\mathcal{K})(\hat{\theta} - \theta) \leq \frac{\bar{\beta}\alpha}{2}\|\delta\| + \frac{\bar{\beta}}{2a}\|\theta - \hat{\theta}\| \tag{39}$$

Therefore, using $C = \sup_{t\geq 0}\|u_i(t)\|$,

$$ {}_0^C D_t^\alpha \bar{V} \leq \sum_{i=1}^N -\left(\beta\varkappa^2 - M^*\varkappa^2 C - \frac{\bar{\beta}}{2a}\right)\|\theta - \hat{\theta}\|^2 - \left(\beta_w - \frac{\bar{\beta}a}{2}\right)\|\delta\|^2 \tag{40}$$

In other words, this problem can be formulated in Theorem 2 to obtain the consensus control law for the multi-agent system.

Theorem 2.

Consider the nonlinear fractional order multi-agent system (3) with control law (26). If there exist $Y \in \mathbb{R}^n$ such that (29) holds, then the consensus control with $\mathcal{K} \in \mathbb{R}^n$, $\mathcal{L} \in \mathbb{R}^m$ can be achieved through the lemma 3 based on the Lyapunov Theorm if the following conditions are satisfied:

$$0 < \beta_w - \frac{\bar{\beta}a}{2}, 0 < \beta\varkappa^2 - M^*\varkappa^2 C - \frac{\bar{\beta}}{2a}, \quad C > \sup_{t\geq 0}\|u_i(t)\| \tag{41}$$

The control signals for all agents must be assumed to be bounded due to the term $\frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(x)g(x) - \frac{{}_0^C\partial_t^\alpha \mu}{{}_0^C\partial_t^\alpha x}(\hat{x})g(\hat{x})$ in the difference $z - \hat{z}$ dynamics. While the specific procedure for designing consensus control for nonlinear systems that admit exact feedback linearization is not detailed here, the procedure presented in (Rehák & Lynnyk, 2021b) can be used for linearized multi-agent systems.

4 SIMULATION AND RESULT

In this section, we will first examine a simple example of feedback linearization for nonlinear system, followed by a more complex example (Rehák & Lynnyk, 2021a). The following is the first example.

$$\begin{aligned} {}_0^C D_t^\alpha x_1 &= x_1^2 + x_1x_2 \\ {}_0^C D_t^\alpha x_2 &= 3x_1^2x_2 + x_1 + u \\ y &= x_1 \end{aligned} \tag{42}$$

It is chosen

$${}_0^C D_t^\alpha y = {}_0^C D_t^\alpha x_1 = x_1^2 + x_1x_2 \tag{43}$$

This means that it needs to be derived again since the control signal has not shown up yet.

$$\begin{aligned} &{}_0^C D_t^\alpha({}_0^C D_t^\alpha y) \\ &= 2x_1\,{}_0^C D_t^\alpha x_1 + {}_0^C D_t^\alpha x_1 x_2 + x_1\,{}_0^C D_t^\alpha x_2 \\ &= (2x_1 + x_2)(x_1^2 + x_1x_2) \\ &+ x_1(3x_1^2x_2 + x_1 + u) \end{aligned} \tag{44}$$

It is important to note that the system has a relative degree of 2, meaning $\rho = 2$ and is defined on $D = \{x \in \mathbb{R}^2 : x_1 \neq 0\}$. Hence, under these circumstances, considering z_1 and z_2, we have: $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1^2 + x_1x_2 \end{bmatrix}$ where $L_f h(x) = \frac{{}_0^C\partial_t^\alpha h}{{}_0^C\partial_t^\alpha x} f(x) = [1 \quad 0]\begin{bmatrix} x_1^2 + x_1x_2 \\ \blacksquare \end{bmatrix} = x_1^2 + x_1x_2$. Therefore, in this case, the result is as follows:

$${}_0^C D_t^\alpha z_1 = {}_0^C D_t^\alpha x_1 = x_1^2 + x_1x_2 = z_2$$

where

$$\begin{aligned} {}_0^C D_t^\alpha z_2 &= 2x_1\,{}_0^C D_t^\alpha x_1 + {}_0^C D_t^\alpha x_1 x_2 + x_2\,{}_0^C D_t^\alpha x_2 \\ &= (2x_1 + x_2)(x_1^2 + x_1x_2) + x_1(3x_1^2x_2 + x_1 + u) \\ &= (2x_1 + x_2)(x_1^2 + x_1x_2) + 3x_1^3x_2 + 3x_1^3 + x_2 + x_1^2 + x_1u \end{aligned} \tag{45}$$

In this case, the consensus control law used to cancel out the nonlinear term is defined by the following relationship.

$$u = -(2x_1 + x_2)(x_1^2 + x_1x_2) - 3x_1^3x_2 - x_1^2 + \frac{1}{x_1}v \tag{46}$$

Due to page limitations, the optimal feedback problem in the z space, where ${}_0^C D_t^\alpha z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} b$, can be solved using methods such as pole placement with $\{p_1 = -2, p_2 = -3\}$. This approach allows for obtaining the optimal feedback more efficiently, with $\{k_1 = 6, k_2 = 5\}$.

$$u = -(2x_1 + x_2)(x_1^2 + x_1x_2) - 3x_1^3x_2 - x_1^2 - k_1 - k_2(x_1 + x_2) \tag{47}$$

It can be easily regulated by adding the control signal (47) to the system equation (42).

Example 2.

Five networked agents, coupled in a ring topology are selected as an example system which depicted in figure 1.

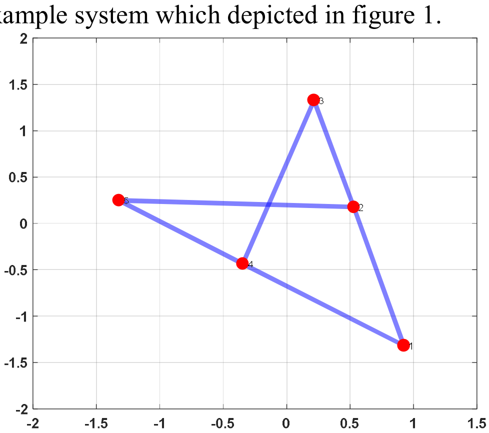

Figure 1: communication between agents

The following equations control the system of each agent (the agent's number is indicated in the introduction).

$$\begin{aligned} {}_0^C D_t^\alpha x_1 &= x_2 \\ {}_0^C D_t^\alpha x_2 &= -(x_1 + x_1^3)e^{x_1} - 0.1x_2 + u \\ y &= x_1 \end{aligned} \tag{48}$$

Where $\bar{A}$ and b are selected and

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{49}$$

These definitions imply that function μ is determined from equations (12),(13), where

$$\mu(x_1, x_2) = \begin{bmatrix} [0.24x_1 - 0.26x_2 - 0.17x_1^2 \\ +0.01x_1x_2 - 0.12x_2^2 - 0.15x_1^3 + \\ 0.14x_1^2x_2 - 0.055x_2^2 + 0.08x_2^3] \\ [0.2x_1 - 0.11x_2 - 0.04x_1^2 - \\ 0.04x_1^2 + 0.027x_1x_2 - 0.01x_2^2 - \\ 0.048x_1^3 + 0.04x_1^2x_2 - 0.03x_1x_2^2 \\ +0.012x_2^3] \end{bmatrix} \tag{50}$$

also with considering $x' = (x_1', x_2')^T$

$$L(x_1', x_2') = \begin{bmatrix} [0.027 + 024x_1' - \\ 0.14x_2' - 0.34x_1'^2 \\ 0.16x_1'x_2' + 0.17x_2'^2] \\ [0.1 - 0.06x_1' + 0.06x_2' \\ 0.1 - 0.06x_1' + 0.06x_2' \\ +0.033x_1'x_2' + 0.016x_2'^2] \end{bmatrix} \tag{51}$$

A linear system is obtained when an agent is precisely feedback linearized with $\theta_1 = x_1, \theta_2 = x_2$ and $\nu = u + (-x_1 - x_1^3)e^{x_1} + x_1$. $\mathcal{K} = [0.415 \quad -1.256]$, is the outcome of the linear quadratic control design. This leads us to the application of the previously discussed observer design to multi-agent management. The i^{th} agent, with its states indicated as $x_i = (x_{i,1}, x_{i,2})^T$, is equipped with an observer:

$$\begin{aligned} {}_0^C D_t^\alpha \hat{x}_{1,i} &= \hat{x}_{i,2} + \mathcal{L}_1(\hat{x}_{i,1}, \hat{x}_{i,2})(x_{i,1} - \hat{x}_{i,1}) \\ {}_0^C D_t^\alpha \hat{x}_{2,i} &= -(\hat{x}_{i,1} + \hat{x}_{i,1}^3)e^{\hat{x}_{i,1}} - 0.1\hat{x}_{i,2} \\ &\quad + \mathcal{L}_2(\hat{x}_{i,1}, \hat{x}_{i,2})(x_{i,1} - \hat{x}_{i,1}) + u_i \end{aligned} \tag{52}$$

Given that $\theta_i = x_i$, the control v_i intended to serve as a consensus control for a linearized agent is:

$$\begin{aligned} v_i &= \mathcal{K}(\hat{\theta}_{i+1} + \hat{\theta}_{i-1} - 2\hat{\theta}_i) \\ &= \mathcal{K}(\hat{x}_{i+1} + \hat{x}_{i-1} - 2\hat{x}_i) \\ &\text{if } i = 2,3,4 \\ v_1 &= \mathcal{K}(\hat{\theta}_2 + \hat{\theta}_5 - 2\hat{\theta}_1) = \\ &\mathcal{K}(\hat{x}_2 + \hat{x}_5 - 2\hat{x}_1) \\ v_5 &= \mathcal{K}(\hat{\theta}_1 + \hat{\theta}_4 - 2\hat{\theta}_5) \\ &= \mathcal{K}(\hat{x}_1 + \hat{x}_4 - 2\hat{x}_5) \end{aligned} \tag{53}$$

in this case, u_i is given by:

$$u_i = v_i - (-\hat{x}_{i,1} - \hat{x}^3{}_{i,3})e^{\hat{x}_{i,1}} - \hat{x}_{i,1} \tag{54}$$

Indeed, further explanations, including the types and conditions of feedback linearization, are provided in, (Khalil, 2002) and cannot be fully presented here. The initial conditions for the state $x_{i,2}$ were chosen as $\{-0.0452, 0.6830, 0.2888, 0.1661, 0.0331\}$, While all observer initial conditions were set to zero, the results are shown in the following figures. Figure 2 depicts the convergence of the first states of the agents.

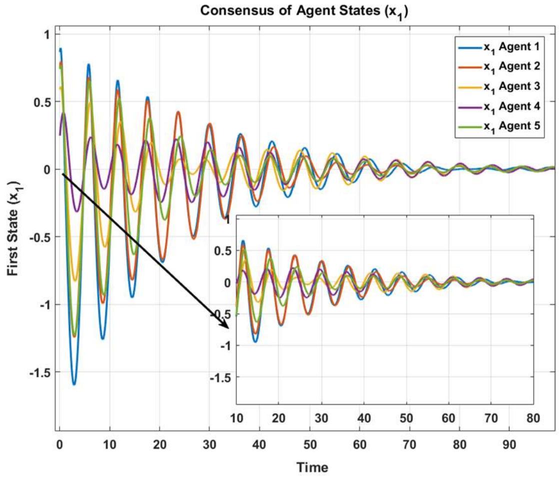

Figure 2: convergence of the agents' first states

In Figure 3, the consensus of the states $x_{i,2}$ is displayed for each agent. It is clear that convergence has been achieved.

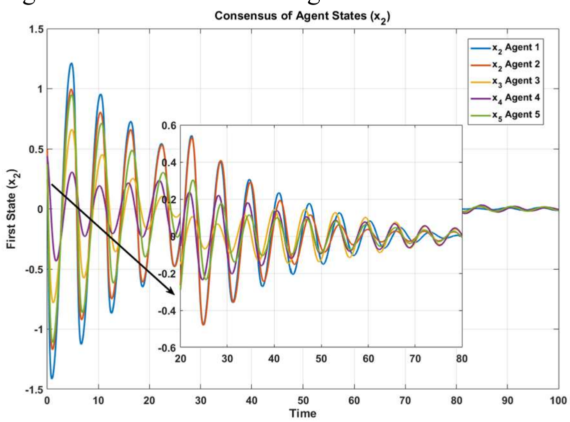

Figure 3: convergence of the agents' second states

In figure 4, the estimation error of the first state of the agent is shown. It is evident that the designed control system exhibits excellent performance.

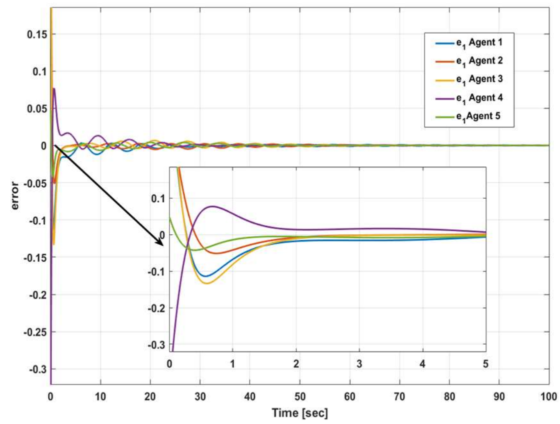

Figure 4: convergence of the agents' first states

Figure 5 shows the state estimation error for the second state, where $e = x_{2,i} - \hat{x}_{2,i}$ for all agents, showing that the estimate converges faster than the system state, indicating that the agents' coupling has been achieved.

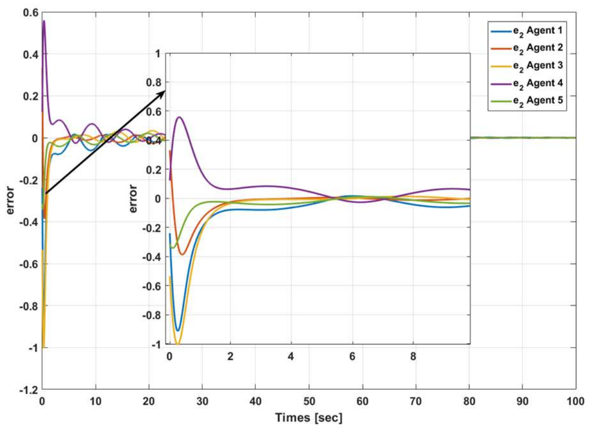

Figure 5: consensus error of second states and its estimation

5 CONCLUSIONS

A novel algorithm for coordinating a multi-agent system with nonlinear agents has been developed, utilizing dynamic output feedback. The observer design is based on an adaptation of the Luenberger approach, ensuring robust performance. The study demonstrates that consensus can be achieved when both the controller and observer satisfy specific conditions outlined in the paper. These findings are further validated through a practical example that illustrates the effectiveness of the proposed method. Future work will focus on extending this research to address the challenges posed by delayed measurements of agent states, which are common in real-world applications. Incorporating delay compensation strategies into the framework will provide a more comprehensive solution, ensuring the algorithm's applicability in a wider range of scenarios where communication and processing delays are inevitable. Moreover, exploring the robustness of the algorithm against disturbances and uncertainties could enhance its adaptability to complex, dynamic environments.

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