

The Consensus Task in a Network with Positive Strictly Metzler Agents

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Abstract—This article deals with the consensus principle in a network of identical linear agents with strictly Metzler dynamics, when the considered communication topology is represented by an undirected graph. To guarantee the functionality of the observer-type consensus protocol, the design problem is parameterized using a structure of linear matrix inequalities, which reflects the principle of diagonal stabilization and singular decomposition of the Laplacian of the network topology graph. Design is clearly illustrated with an example.

I. INTRODUCTION

Cooperative systems control has received considerable attention due to its applications in many industries, [1], [2], [3], [4] and multiagent systems have become interesting areas in intelligent resource management and agent-oriented technologies [5], [6], [7] with a specific focus on consensus, coordination and synchronization problems. Some of these industries belong to the class of positive systems, [8], [9], [10] and, restricting themselves to the Metzler structure of the dynamics matrices of their models, are often referred to as Metzler systems, [11].

For practical reasons, full consensus protocols in multiagent networks are replaced by protocols using distributed state estimation via observers, [12], [13], [14], where convergence of consensus algorithms requires only a weak form of connectivity of the communication graph, [15], [16], [17].

This paper deals with the solution of the consensus problem in linear strictly Metzler agents. Assuming that N is the number of agents described by the same linear Metzler model, the main results are the parameter design conditions that ensure the stability of the protocol and reflect the constraints imposed on positive systems. The design conditions are formulated using the linear matrix inequalities (LMIs), and the main design constraints are considered to be diagonal stability and the parametric constraints.

The basic preliminaries are presented in Section II and the consensus problem is solved in Section III. The LMIs for parameters design are formulated in Section IV. Section V shows the applicability of the method with an example, and section VI summarizes the results.

Throughout the paper $\mathbf{X} \prec 0$ conveys that a real matrix \mathbf{X} is symmetric and negative definite, notations \mathbf{x}^T (\mathbf{X}^T) identifies the transpose of the vector (matrix), \mathbf{I}_n indicates the n -th order unit matrix, $\text{diag}[\cdot]$ enters up a block diagonal matrix, \otimes denotes the Kronecker product of matrices, \mathbb{R}^n denotes the n -dimensional Euclidean space, $\mathbb{R}_+^{n \times r}$ ($\mathbb{R}_{-+}^{n \times n}$) refers to the set of real nonnegative (Metzler) matrices.

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II. BASIC PRELIMINARIES

Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph of order N , where $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$ is the set of edges and $\mathcal{A} = \{\alpha_{ij}\} \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix, [18], [19]. The adjacency elements satisfy the conditions $\alpha_{ii} = 0$, $\alpha_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. If the graph is undirected, $\alpha_{ij} = \alpha_{ji}$ and \mathcal{A} is symmetric.

The Laplacian matrix \mathcal{L} to the graph \mathcal{G} is defined as $\mathcal{L} = \{l_{ij}\} \in \mathbb{R}^{N \times N}$, $l_{ij} = -\alpha_{ij}$, $i \neq j$ and $l_{ii} = \sum_{j \neq i} \alpha_{ij}$.

For an undirected graph, both the adjacency matrix and the Laplacian matrix are symmetric, and the row and column sums of the Laplacian matrix are all zero, which means that for an undirected graph, the Laplacian matrix is singular and diagonally dominant, [20], [21].

The Kronecker product \otimes of matrices $\mathbf{X} = \{x_{n,m}\} \in \mathbb{R}^{n \times m}$ and $\mathbf{Y} \in \mathbb{R}^{w \times v}$ is the operation

$$\mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} x_{11}\mathbf{Y} & \cdots & x_{1m}\mathbf{Y} \\ \vdots & \ddots & \vdots \\ x_{n1}\mathbf{Y} & \cdots & x_{nm}\mathbf{Y} \end{bmatrix}, \quad \mathbf{X} \otimes \mathbf{Y} \in \mathbb{R}^{nw \times mv} \quad (1)$$

and for matrices $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ of appropriated dimensions and a scalar a satisfies the basic properties, [22],

$$\begin{aligned} (\mathbf{X} \otimes \mathbf{Z})(\mathbf{Y} \otimes \mathbf{W}) &= \mathbf{X}\mathbf{Y} \otimes \mathbf{Z}\mathbf{W} \\ \mathbf{X} \otimes (\mathbf{Y} + \mathbf{Z}) &= \mathbf{X} \otimes \mathbf{Y} + \mathbf{X} \otimes \mathbf{Z} \\ (\mathbf{X} \otimes \mathbf{Z})^T &= \mathbf{X}^T \otimes \mathbf{Z}^T \\ a\mathbf{X} &= a \otimes \mathbf{X} \end{aligned} \quad (2)$$

Consider multiagent systems with a set of N internally positive and linear strictly Metzler agents, described for $i = 1, \dots, N$ by the realizations on the parameters $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ as

$$\dot{\mathbf{q}}_i(t) = \mathbf{A}\mathbf{q}_i(t) + \mathbf{B}\mathbf{u}_i(t), \quad \mathbf{y}_i(t) = \mathbf{C}\mathbf{q}_i(t) \quad (3)$$

where the vectors $\mathbf{q}_i(t) \in \mathbb{R}_+^n$, $\mathbf{y}_i(t) \in \mathbb{R}_+^m$ are positive for $\mathbf{q}(0) > 0$, matrices $\mathbf{B} \in \mathbb{R}_+^{n \times r}$, $\mathbf{C} \in \mathbb{R}_+^{m \times n}$ are nonnegative and $\mathbf{A} \in \mathbb{M}_{-+}^{n \times n}$ is a strictly Metzler matrix, whilst the elements $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ must satisfy the constraints

$$a_{ii} < 0, \quad a_{ij} > 0, i \neq j, \quad \forall i, j \in \langle 1, n \rangle \quad (4)$$

In order to perform the analysis of the system, basic constraints must be reflected, i.e., since (3) is only diagonally stabilized, [23], the design conditions based on LMIs must be formulated using diagonal positive definite matrix variables, and the parametric constraints (4) must be adapted to diagonal LMI structures.

For the i th system (3), the structure of the observer is

$$\begin{aligned} \dot{\mathbf{q}}_{ei}(t) &= \mathbf{A}\mathbf{q}_{ei}(t) + \mathbf{B}\mathbf{u}_i(t) + \mathbf{J}(\mathbf{y}_i(t) - \mathbf{y}_{ei}(t)) \\ \mathbf{y}_{ei}(t) &= \mathbf{C}\mathbf{q}_{ei}(t) \end{aligned} \quad (5)$$

where $\mathbf{q}_{ei}(t) \in \mathbb{R}_+^n$, $\mathbf{y}_{ei}(t) \in \mathbb{R}_+^m$ and $\mathbf{J} \in \mathbb{R}_+^{n \times m}$ is nonnegative.

III. CONSENSUS PROBLEM

The goal is to design a consensus protocol $\mathbf{u}_i(t)$ of a multi-agent system using the estimated values of the agents' state vectors, where the following form of consensus protocol is proposed

$$\mathbf{u}_i(t) = \mathbf{K} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) \quad (6)$$

where \mathcal{N}_i is the neighborhood of the i -th agent, $\mathbf{K} \in \mathbb{R}_+^{r \times n}$ is the protocol gain matrix and α_{ij} represent the neighborhood elements of the adjacency matrix $\mathcal{A} = [\alpha_{ij}]_{i,j=1}^N \in \mathbb{R}^{N \times N}$.

Connecting the defined models (5) and (3) using the settings

$$\mathbf{e}_i(t) = \mathbf{q}_i(t) - \mathbf{q}_{ei}(t) \quad (7)$$

leads to the dynamic error equations

$$\begin{aligned} \dot{\mathbf{e}}_i(t) &= \mathbf{A}\mathbf{q}_i(t) + \mathbf{B}\mathbf{u}_i(t) - \\ &\quad - \mathbf{A}\mathbf{q}_{ei}(t) - \mathbf{B}\mathbf{u}_i(t) - \mathbf{J}\mathbf{C}(\mathbf{q}_i(t) - \mathbf{q}_{ei}(t)) \quad (8) \\ &= (\mathbf{A} - \mathbf{J}\mathbf{C})\mathbf{e}_i(t) \end{aligned}$$

and based on the Kronecker product property, the entire structure have the form

$$\dot{\mathbf{e}}(t) = (\mathbf{I}_N \otimes \mathbf{A}_e)\mathbf{e}(t) \quad (9)$$

where

$$\mathbf{e}(t) = [e_1^T(t) \cdots e_N^T(t)]^T \in \mathbb{R}^{Nn} \quad (10)$$

$$\mathbf{A}_e = \mathbf{A} - \mathbf{J}\mathbf{C} \quad (11)$$

If the synchronizing state error vector is defined as

$$\boldsymbol{\delta}_i(t) = \mathbf{q}_i(t) - \frac{1}{N} \sum_{j=1}^N \mathbf{q}_j(t) = \mathbf{q}_i(t) - \bar{\mathbf{q}}(t) \quad (12)$$

where

$$\bar{\mathbf{q}}(t) = \frac{1}{N} \sum_{j=1}^N \mathbf{q}_j(t) \quad (13)$$

then dynamics of the synchronization error vector is

$$\begin{aligned} \dot{\boldsymbol{\delta}}_i(t) &= \dot{\mathbf{q}}_i(t) - \dot{\bar{\mathbf{q}}}_i(t) \\ &= \mathbf{A}(\mathbf{q}_i(t) - \bar{\mathbf{q}}_i(t)) + \mathbf{B}\mathbf{K} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) - \\ &\quad - \frac{1}{N} \sum_{j \in \mathcal{N}_i} \sum_{i=1}^N \mathbf{B}\mathbf{K} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) \\ &= \mathbf{A}\boldsymbol{\delta}_i(t) + \mathbf{B}\mathbf{K} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) \end{aligned} \quad (14)$$

where the last row relation follows from the property of symmetry of undirected topology graph $\alpha_{ij} = \alpha_{ji}$, which implies that

$$\frac{1}{N} \sum_{j \in \mathcal{N}_i} \sum_{i=1}^N \mathbf{B}\mathbf{K} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) = \mathbf{0} \quad (15)$$

Since the following holds

$$\begin{aligned} &= \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_{ei}(t) - \mathbf{q}_{ej}(t)) \\ &= \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_i(t) - \mathbf{e}_i(t) - (\mathbf{q}_j(t) - \mathbf{e}_j(t))) \\ &= \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\mathbf{q}_i(t) - \bar{\mathbf{q}}(t) - \mathbf{q}_j(t) + \bar{\mathbf{q}}(t) - \mathbf{e}_i(t) + \mathbf{e}_j(t)) \\ &= \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\boldsymbol{\delta}_i(t) - \boldsymbol{\delta}_j(t) - \mathbf{e}_i(t) + \mathbf{e}_j(t)) \end{aligned} \quad (16)$$

then

$$\begin{aligned} \dot{\boldsymbol{\delta}}_i(t) &= \mathbf{A}\boldsymbol{\delta}_i(t) + \\ &\quad + \mathbf{B}\mathbf{K} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\boldsymbol{\delta}_i(t) - \boldsymbol{\delta}_j(t) - \mathbf{e}_i(t) + \mathbf{e}_j(t)) \end{aligned} \quad (17)$$

and for the obtained network dynamics this gives equivalently

$$\dot{\boldsymbol{\delta}}(t) = (\mathbf{I}_N \otimes \mathbf{A})\boldsymbol{\delta}(t) + (\mathcal{L} \otimes \mathbf{B}\mathbf{K})(\boldsymbol{\delta}(t) - \mathbf{e}(t)) \quad (18)$$

where

$$\boldsymbol{\delta}(t) = [\boldsymbol{\delta}_1^T(t) \cdots \boldsymbol{\delta}_N^T(t)]^T \in \mathbb{R}^{Nn} \quad (19)$$

Using the property of the Kronecker product, the extended dynamics has the form

$$\begin{bmatrix} \dot{\boldsymbol{\delta}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_N \otimes \mathbf{A} + \mathcal{L} \otimes \mathbf{B}\mathbf{K} & -\mathcal{L} \otimes \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{I}_N \otimes \mathbf{A}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}(t) \\ \mathbf{e}(t) \end{bmatrix} \quad (20)$$

or, written compactly,

$$\dot{\mathbf{z}}(t) = \mathbf{A}^\circ \mathbf{z}(t) \quad (21)$$

where

$$\mathbf{A}^\circ = \begin{bmatrix} \mathbf{I}_N \otimes \mathbf{A} + \mathcal{L} \otimes \mathbf{B}\mathbf{K} & -\mathcal{L} \otimes \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{I}_N \otimes \mathbf{A}_e \end{bmatrix}, \mathbf{z}(t) = \begin{bmatrix} \boldsymbol{\delta}(t) \\ \mathbf{e}(t) \end{bmatrix} \quad (22)$$

Remark 1: Since \mathcal{L} of an undirected graph is symmetric positive semi-definite, singular value decomposition of \mathcal{L} takes the form, [24],

$$\mathbf{M}^T \mathcal{L} \mathbf{M} = \mathbf{S} \quad (23)$$

where

$$\mathbf{M} = [\mathbf{m}_1 \mathbf{m}_2 \cdots \mathbf{m}_N], \mathbf{S} = \text{diag} [s_1 \cdots s_{N-1} 0] \quad (24)$$

and with the singular value set written in an descending order as defined in \mathbf{S}

$$s_1 \geq s_2 \geq \cdots \geq s_{N-1} \geq 0 \quad (25)$$

the orthonormal vectors $\mathbf{m}_h, h = 1, \dots, N$, span the matrix \mathbf{M} such that

$$\mathbf{M}^T \mathbf{M} = \mathbf{M} \mathbf{M}^T = \mathbf{I}_N, \mathbf{M}^{-1} = \mathbf{M}^T \quad (26)$$

$$\mathcal{L} = \mathbf{M} \mathbf{S} \mathbf{M}^T \quad (27)$$

The following theorem is proposed for the application of such a set of singular values to guarantee the functionality and stability of the observer-type consensus protocol.

Theorem 1: If for the error (21) of the positive multiagent system (3) and its Laplacian matrix \mathcal{L} (23) there exists with respect to all $j = 1, \dots, N-1$ positive definite diagonal matrices $\mathbf{P}_1, \mathbf{P}_A \in \mathbb{R}^{n \times n}$ such that following LMI constraints are feasible

$$\mathbf{P}_1 \succ 0, \quad \mathbf{P}_A \succ 0 \quad (28)$$

$$\Xi_j = \begin{bmatrix} \Xi_{j11} & * \\ -s_j \mathbf{P}_1 \mathbf{B} \mathbf{K} & \Theta_{22} \end{bmatrix} \prec 0 \quad (29)$$

$$\Theta_{22} = \Theta_N = \mathbf{P}_2(\mathbf{A} - \mathbf{J} \mathbf{C}) + (\mathbf{A}^\top - \mathbf{C}^\top \mathbf{J}^\top) \mathbf{P}_2 \prec 0 \quad (30)$$

$$\Sigma_N = \mathbf{P}_1 \mathbf{A} + \mathbf{A}^\top \mathbf{P}_1 \prec 0 \quad (31)$$

where

$$\Xi_{j11} = \mathbf{P}_1(\mathbf{A} + s_j \mathbf{B} \mathbf{K}) + (\mathbf{A}^\top + s_j \mathbf{K}^\top \mathbf{B}^\top) \mathbf{P}_1 \quad (32)$$

then the consensus problem for (3) is solvable under the consensus protocol (6).

Hereafter, * is a symmetric entry in a symmetric matrix.

Proof: In order to respect the principle of diagonal stabilization for the error system (21), the Lyapunov function candidate is constructed

$$v(\mathbf{z}(t)) = \mathbf{z}^\top(t) \mathbf{P}^\circ \mathbf{z}(t) > 0 \quad (33)$$

where for $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{R}^{n \times n}$

$$\mathbf{P}^\circ = \text{diag} [\mathbf{I}_N \otimes \mathbf{P}_1 \quad \mathbf{I}_N \otimes \mathbf{P}_2] \succ 0 \quad (34)$$

$$\mathbf{P}_1 = \text{diag} [p_{11} \dots p_{1n}] \succ 0 \quad (35)$$

$$\mathbf{P}_2 = \text{diag} [p_{21} \dots p_{2n}] \succ 0$$

The corresponding time derivative of $v(\mathbf{z}(t))$ along the solutions (33) is

$$\dot{v}(\mathbf{z}(t)) = \dot{\mathbf{z}}^\top(t) \mathbf{P}^\circ \mathbf{z}(t) + \mathbf{z}^\top(t) \mathbf{P}^\circ \dot{\mathbf{z}}(t) < 0 \quad (36)$$

and using the composed vector defined in (22) then

$$\begin{aligned} & \begin{bmatrix} \dot{\delta}^\top(t) & \dot{e}^\top(t) \end{bmatrix} \begin{bmatrix} \mathbf{I}_N \otimes \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \otimes \mathbf{P}_2 \end{bmatrix} \begin{bmatrix} \delta(t) \\ e(t) \end{bmatrix} + \\ & + \begin{bmatrix} \delta^\top(t) & e^\top(t) \end{bmatrix} \begin{bmatrix} \mathbf{I}_N \otimes \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \otimes \mathbf{P}_2 \end{bmatrix} \begin{bmatrix} \dot{\delta}(t) \\ \dot{e}(t) \end{bmatrix} \end{aligned} \quad (37)$$

< 0

which implies

$$\begin{aligned} & \dot{\delta}^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1) \delta(t) + \delta^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1) \dot{\delta}(t) + \\ & + \dot{e}^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2) e(t) + e^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2) \dot{e}(t) \end{aligned} \quad (38)$$

< 0

Using expressions (9), (18) give the possibility to rewrite (38) as

$$\begin{aligned} & \delta^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1 \mathbf{A} + \mathcal{L} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \delta(t) + \\ & + \delta^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1 \mathbf{A} + \mathcal{L} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \delta(t) + \\ & + e^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2 \mathbf{A}_e) e(t) + e^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2 \mathbf{A}_e)^\top e(t) - \\ & - e^\top(t) (\mathcal{L} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \delta(t) - \delta^\top(t) (\mathcal{L} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) e(t) \end{aligned} \quad (39)$$

< 0

The relations (26), (27) allow to write

$$\begin{aligned} & \delta^\top(t) (\mathbf{M} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{M} \mathbf{S} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \delta(t) + \\ & + \delta^\top(t) (\mathbf{M} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{M} \mathbf{S} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \delta(t) \\ & = \delta^\top(t) \mathbf{N}^\top (\mathbf{I}_n \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \mathbf{N} \delta(t) + \\ & + \delta^\top(t) \mathbf{N}^\top (\mathbf{I}_n \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \mathbf{N} \delta(t) \end{aligned} \quad (40)$$

$$\begin{aligned} & \delta^\top(t) (\mathbf{M} \mathbf{S} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) e(t) + \\ & + e^\top(t) (\mathbf{M} \mathbf{S} \mathbf{M}^\top \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \delta(t) \\ & = \delta^\top(t) \mathbf{N}^\top (\mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \mathbf{N} e(t) + \\ & + e^\top(t) \mathbf{N}^\top (\mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \mathbf{N} \delta(t) \end{aligned} \quad (41)$$

$$\begin{aligned} & e^\top(t) (\mathbf{M} \mathbf{M}^\top \otimes \mathbf{P}_2 \mathbf{A}_e) e(t) + \\ & + e^\top(t) (\mathbf{M} \mathbf{M}^\top \otimes \mathbf{P}_2 \mathbf{A}_e)^\top e(t) \\ & = e^\top(t) \mathbf{N}^\top (\mathbf{I}_n \otimes \mathbf{P}_2 \mathbf{A}_e) \mathbf{N} e(t) + \\ & + e^\top(t) \mathbf{N}^\top (\mathbf{I}_n \otimes \mathbf{P}_2 \mathbf{A}_e)^\top \mathbf{N} e(t) \end{aligned} \quad (42)$$

where

$$\mathbf{N} = \mathbf{M}^\top \otimes \mathbf{I}_N \quad (43)$$

and by defining the following vector variable transformations

$$\begin{aligned} \mathbf{N} \delta(t) &= (\mathbf{M}^\top \otimes \mathbf{I}_N) \delta(t) = \zeta(t) \\ \mathbf{N} e(t) &= (\mathbf{M}^\top \otimes \mathbf{I}_N) e(t) = \varepsilon(t) \end{aligned} \quad (44)$$

(39) can be rewritten as

$$\begin{aligned} & \zeta^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \zeta(t) + \\ & + \zeta^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_1 \mathbf{A} + \mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \zeta(t) + \\ & + \varepsilon^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2 \mathbf{A}_e) \varepsilon(t) + \varepsilon^\top(t) (\mathbf{I}_N \otimes \mathbf{P}_2 \mathbf{A}_e)^\top \varepsilon(t) - \\ & - \zeta^\top(t) (\mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K}) \varepsilon(t) - \varepsilon^\top(t) (\mathbf{S} \otimes \mathbf{P}_1 \mathbf{B} \mathbf{K})^\top \zeta(t) \end{aligned} \quad (45)$$

< 0

Thus, in view of (24) and (45), it yields that

$$\begin{aligned} & \sum_{j=1}^{N-1} \zeta_j^\top(t) (\mathbf{P}_1(\mathbf{A} + s_j \mathbf{B} \mathbf{K}) + (\mathbf{A} + s_j \mathbf{B} \mathbf{K})^\top \mathbf{P}_1) \zeta_j(t) - \\ & - \sum_{j=1}^{N-1} (\zeta_j^\top(t) s_j \mathbf{P}_1 \mathbf{B} \mathbf{K} \varepsilon_j(t) + \varepsilon_j^\top(t) s_j \mathbf{K}^\top \mathbf{B}^\top \mathbf{P}_1 \zeta_j(t)) + \\ & + \sum_{j=1}^N \varepsilon_j^\top(t) (\mathbf{P}_2 \mathbf{A}_e + \mathbf{A}_e^\top \mathbf{P}_2) \varepsilon_j(t) \end{aligned} \quad (46)$$

Connecting the variable vector components then (46) gives the stability condition as

$$\sum_{j=1}^{N-1} \xi_j^\top(t) \Xi_j \xi_j(t) + \varepsilon_N^\top(t) \Theta_N \varepsilon_N(t) + \zeta_N^\top(t) \Sigma_N \zeta_N(t) < 0 \quad (47)$$

using (29)–(32) and the proof is completed. \blacksquare

Theorem 3: Given a strictly Metzler matrix $\mathbf{A} \in \mathbb{M}_{-+}^{n \times n}$, a nonnegative $\mathbf{B} \in \mathbb{R}_{+}^{n \times r}$ and a stable strictly Metzler matrix $\mathbf{A}_e \in \mathbb{M}_{-+}^{n \times n}$ then the consensus for (3) is stable with a strictly Metzler and Hurwitz matrix $\mathbf{A}_c \in \mathbb{M}_{-+}^{n \times n}$ if there exist positive definite diagonal matrices $\mathbf{Q}_1, \mathbf{R}_k \in \mathbb{R}_{+}^{n \times n}$ such that for $h = 1, \dots, n-1, j = 1, \dots, N-1$ can be find any feasible solution of the following set of LMIs

$$\mathbf{Q}_1 \succ 0, \mathbf{R}_k \succ 0 \quad (69)$$

$$\mathbf{A}_{\Theta}(v, v) \mathbf{Q}_1 + s_j \sum_{k=1}^m \mathbf{B}_k \mathbf{R}_k \prec 0 \quad (70)$$

$$\mathbf{T}^h \mathbf{A}_{\Theta}(v, v+h) \mathbf{T}^{h\top} \mathbf{Q}_1 + s_j \sum_{k=1}^m \mathbf{T}^h \mathbf{B}_k \mathbf{T}^{h\top} \mathbf{R}_k \succ 0 \quad (71)$$

$$\mathbf{A} \mathbf{Q}_1 + \mathbf{Q}_1 \mathbf{A}^{\top} + s_j \sum_{k=0}^m (\mathbf{B}_k \mathbf{l}_n \mathbf{l}_n^{\top} \mathbf{R}_k + \mathbf{R}_k \mathbf{l}_n \mathbf{l}_n^{\top} \mathbf{B}_k) \prec 0 \quad (72)$$

where the strictly positive $\mathbf{K} \in \mathbb{R}_{+}^{n \times m}$ is computed as

$$\mathbf{K}_k = \mathbf{R}_k \mathbf{Q}_1^{-1}, \mathbf{k}_k^{\top} = \mathbf{l}_n^{\top} \mathbf{K}_k, \mathbf{K} = [\mathbf{k}_1 \ \dots \ \mathbf{k}_r]^{\top} \quad (73)$$

The proof can be omitted because it also follows from the duality principle.

V. ILLUSTRATIVE EXAMPLE

To present new results it is considered that the system is given with the matrices with their parametrization

$$\mathbf{A} = \begin{bmatrix} -3.380 & 2.986 & 5.520 & 1.9620 \\ 0.581 & -4.708 & 1.631 & 0.3490 \\ 1.067 & 2.146 & -8.780 & 3.3980 \\ 0.048 & 1.401 & 0.471 & -2.7700 \end{bmatrix}$$

$$\mathbf{A}_{\Theta}(v, v) = \text{diag} [-3.380 \ -4.708 \ -8.780 \ -2.770]$$

$$\mathbf{A}_{\Theta}(v+1, v) = \text{diag} [0.581 \ 2.146 \ 0.471 \ 1.962]$$

$$\mathbf{A}_{\Theta}(v+2, v) = \text{diag} [1.067 \ 1.401 \ 5.520 \ 0.349]$$

$$\mathbf{A}_{\Theta}(v+3, v) = \text{diag} [0.048 \ 2.986 \ 1.631 \ 3.398]$$

$$\mathbf{A}_{\Theta}(v, v+1) = \text{diag} [2.986 \ 1.631 \ 3.398 \ 0.048]$$

$$\mathbf{A}_{\Theta}(v, v+2) = \text{diag} [5.520 \ 0.349 \ 1.067 \ 1.401]$$

$$\mathbf{A}_{\Theta}(v, v+3) = \text{diag} [1.962 \ 0.581 \ 2.146 \ 0.471]$$

$$\mathbf{B}^{\top} = \begin{bmatrix} 0.0200 & 0.0284 & 0.0057 & 0.0057 \\ 0.0094 & 0.0101 & 0.0157 & 0.0085 \end{bmatrix}$$

$$\mathbf{B}_1 = \text{diag} [0.0200 \ 0.0284 \ 0.0057 \ 0.0057]$$

$$\mathbf{B}_2 = \text{diag} [0.0094 \ 0.0101 \ 0.0157 \ 0.0085]$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}_1 = \text{diag} [0 \ 1 \ 1 \ 0] \\ \mathbf{C}_2 = \text{diag} [0 \ 0 \ 0 \ 1]$$

and for the network with $N = 4$ the Laplacian matrix and its singular values are

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{S} = \text{diag} [4 \ 3 \ 1 \ 0]$$

Using SeDuMi Toolbox, [28], it can be obtained the LMI variable from (54)–(57) as

$$\mathbf{P}_1 = [0.3597 \ 0.4128 \ 0.2230 \ 0.5832] \succ 0$$

$$\mathbf{V}_1 = [0.6470 \ 0.2367 \ 0.1851 \ 0.1360] \succ 0$$

$$\mathbf{V}_2 = [0.3571 \ 0.0675 \ 0.3768 \ 0.3233] \succ 0$$

Then, from (58),

$$\mathbf{J} = \begin{bmatrix} 1.7988 & 0.9927 \\ 0.5734 & 0.1635 \\ 0.8299 & 1.6898 \\ 0.2332 & 0.5543 \end{bmatrix}$$

and

$$\mathbf{A}_e = \mathbf{A} - \mathbf{J} \mathbf{C} = \begin{bmatrix} -3.3800 & 1.1872 & 3.7212 & 0.9693 \\ 0.5810 & -5.2814 & 1.0576 & 0.1855 \\ 1.0670 & 1.3161 & -9.6099 & 1.7082 \\ 0.0480 & 1.1678 & 0.2378 & -3.3243 \end{bmatrix}$$

with the dynamics defined by the eigenvalues set

$$\varrho(\mathbf{A}_e) = \{-1.8772 \ -3.8016 \ -5.5591 \ -10.3577\}$$

Solving (69)–(72) then

$$\mathbf{Q}_1 = [0.7604 \ 0.1839 \ 0.2254 \ 0.1691] \succ 0$$

$$\mathbf{R}_1 = [0.6470 \ 0.2367 \ 0.1851 \ 0.1360] \succ 0$$

$$\mathbf{R}_2 = [0.3571 \ 0.0675 \ 0.3768 \ 0.3233] \succ 0$$

$$\mathbf{K} = \begin{bmatrix} 0.8509 & 1.2872 & 0.8210 & 0.8041 \\ 0.4696 & 0.3671 & 1.6717 & 1.9117 \end{bmatrix}$$

and it can be easily calculated that

$$\mathbf{A}_{e1} = \begin{bmatrix} -3.2942 & 3.1028 & 5.6488 & 2.0985 \\ 0.6967 & -4.5469 & 1.7921 & 0.5179 \\ 1.1159 & 2.1983 & -8.6562 & 3.5366 \\ 0.0833 & 1.4427 & 0.5465 & -2.6867 \end{bmatrix}$$

$$\varrho(\mathbf{A}_{e1}) = \{-0.0134 \ -3.6578 \ -5.4660 \ -10.0467\}$$

$$\mathbf{A}_{e2} = \begin{bmatrix} -3.3156 & 3.0736 & 5.6166 & 2.0644 \\ 0.6678 & -4.5872 & 1.7518 & 0.4757 \\ 1.1037 & 2.1853 & -8.6871 & 3.5019 \\ 0.0745 & 1.4323 & 0.5276 & -2.7075 \end{bmatrix}$$

$$\varrho(\mathbf{A}_{e2}) = \{-0.1218 \ -3.6561 \ -5.4632 \ -10.0563\}$$

$$\mathbf{A}_{e3} = \begin{bmatrix} -3.3585 & 3.0152 & 5.5522 & 1.9961 \\ 0.6099 & -4.6677 & 1.6713 & 0.3912 \\ 1.0792 & 2.1591 & -8.7490 & 3.4326 \\ 0.0568 & 1.4114 & 0.4899 & -2.7492 \end{bmatrix}$$

$$\varrho(\mathbf{A}_{e3}) = \{-0.3403 \ -3.6517 \ -5.4570 \ -10.0755\}$$

Given Laplacian structure \mathcal{L} , a strictly Metzler and Hurwitz matrix \mathbf{A} , and non-negative matrices \mathbf{B} and \mathbf{C} , the presented results confirm that the dynamics matrices of the observer and controller are strictly Metzler and Hurwitz. It can also be seen that the resulted parameter structure significantly affect only the dynamics-dominant eigenvalue of the controller dynamics matrix.

VI. CONCLUDING REMARKS

The main part of the paper was the application of specific design constraints to a consensus protocol for a multiagent system based on estimated values of agent state vectors with strictly Metzler dynamics in a time-invariant communication topology. A distributed observer structure with communication network sensor sharing is used to estimate information about the single agent state, where a is formulated for a set of identical agents described by positive linear Metzler systems. Motivated by the consensus capability for communication constraints, agent dynamics interaction, and undirected graph properties, consensus is studied to project a stable control protocol into optimization under parametric constraints in the LMI structure. By utilizing the SVD representation of the Laplacian matrix, all participating decision variables for the consensus design can have a diagonal structure to reduce design conservatism. The specification in restricting to Metzler dynamics is accompanied by modified insights into the basic protocol properties. Using Lyapunov-based arguments, the design conditions are derived in terms of LMIs to design asymptotic consensus control protocol. Numerical illustrations are given to demonstrate the applicability of the presented formulations.

Such a formulation of the principles provides a potential basis for future research directions focusing on more general cases of agent systems, such as discrete positive agents and ostensible positive agents with time delays, and the design of systems estimating consensus performance based on the network topology.

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