Data-driven predictive control for interconnected systems using terminal ingredients and reachable sets

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***Abstract*— In this paper, we synthesize controllers for linear time-invariant (LTI) systems using collected offline data. We first synthesize control-invariant sets from offline collected data using backward reachable set computations and then propose a data-driven predictive controller equipped with terminal constraints. After formulating the optimal control problem and designing the terminal constraints and costs, we ensure the recursive feasibility of the optimization problem, asymptotic stability of the closed-loop system, and satisfaction of input and state constraints. We develop an overall online algorithm for our approach that does not require the initial state to be included in the control invariant set, guarantees optimal behavior of the system's operation, and does not have Lyapunov constraints that restrict the feasibility region. Furthermore, we extend our developed data-driven control algorithm to stabilize interconnected systems in a decentralized manner, where the satisfaction of a small gain condition is additionally required. We illustrate the effectiveness of our approach through a detailed example.**

***Index Terms*—Data-driven predictive control (DPC), LTI systems, stabilization, reachable sets, set invariance.**

I. Introduction

Recent attention has been directed towards the analysis and design of data-driven systems and controllers. Various approaches have emerged for addressing control problems based on data, and for a comprehensive overview, one can refer to a recent survey [1].

In the context of data-driven control, prior work has tackled data-driven output matching for LTI systems [2], explored data-driven controllability and observability for LTI systems [3], and designed data-driven stabilizing controllers for LTI systems [4]. Data-driven predictive control has been studied in [5]–[7]. Moreover, data-driven methods, including set-based techniques, have been employed to address control problems [8], [9].

This paper contributes to the field by developing data-driven predictive controllers with terminal constraints designed using reachable sets. We approach the problem in two levels of computations: offline and online. We conduct offline calculations to compute using input/state trajectories a contracting set for the closed-loop system. This is accomplished by using concepts of invariance and contractivity [10], allowing us to reformulate the problem as a quadratic program with terminal constraints for efficient solving. Subsequently, we synthesize the sequence of permissible control inputs online, by solving a moving horizon optimization problem based on the current system state and the offline computations.

It is noteworthy that, many existing data-driven control methodologies do not consider state and input constraints, except for predictive controllers as seen in [4], [11], where no explicit contracting set is computed. In these contexts, [12] incorporates reachability analysis within the model predictive control (MPC) framework to compute zonotopes that guarantee robust constraint satisfaction. Additionally, [13] synthesizes control invariant sets using offline data but confines the analysis to systems with static feedback controllers. Similarly, [8], [14] address the stabilization problem through data and set computations. However, they restrict their investigation to designing linear static feedback controllers using an initial set that must be a contracting set for the closed-loop system. Also the predictive controllers in [6] rely on ellipsoidal sets and terminal constraints derived based on a fixed set-point.

In this paper, we focus on the task of designing data-driven controllers for linear time-invariant (LTI) systems, with the objective of stabilizing these systems through the utilization of forward control invariant sets. To compute these sets, we rely on our previous result [15], which synthesize such sets based on the *set-invariance theory* and the *fundamental lemma*, originally developed by Willems and coworkers [16, Theorem 1]. This lemma provides sufficient conditions for the identifiability of LTI systems and offers a data-driven perspective of the system (see Lemma 1). Our approach builds upon our previous result [15] to find, using reachability analysis, a polytopic control invariant set Φ for the system and uses Φ to design a data-driven predictive controller (DPC) with terminal constraints and a terminal cost. Our DPC is composed of an online optimal control problem (OCP) that:

- unlike [15], does not require the initial state to be included in Φ to stabilize the system;
- has a larger feasibility region than the online controllers in [15] since no Lyapunov functions, which decrease at each time step, are added in constraints to the OCP;
- has also been used to control and stabilize an interconnected system.

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The rest of this paper is as follows: Section II gives preliminary concepts, while Section III formulates the problem. In Section IV, we investigate the approaches for solving the control problem for a single system as well as for an interconnected system. Section V provides the algorithm for set-based data-driven control. An illustrative example is presented in Section VI, and the paper concludes with Section VII. We omit proofs due to space limitations, where the extended version of the paper is accessible in [17].

II. Notation and Preliminaries

We use the symbols $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{N}^+$ to represent sets of real numbers, non-negative real numbers, positive real numbers, nonnegative integers, and positive integers, respectively. The column vector with all elements equal to one and of the appropriate dimensions is denoted by $\mathbf{1}$. For $x \in \mathcal{S} \subset \mathbb{R}^n$, $||x||$ denotes its norm and $\mathcal{B}$ denotes the associated unit ball. Given a real matrix $A \in \mathbb{R}^{n\times n}$, $||A||$ is the norm of A induced by the norm $||.||$ For a positive definite matrix $Q > 0$ we denote by $\mathcal{B}_Q$ the ellipsoid $\{x \in \mathbb{R}^n : x^\top Q x \leq 1\}$. A function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$ function and $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and for each fixed $s > 0$ the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to 0$. Finally, $\mathcal{I}_d \in \mathcal{K}_\infty$ denotes the identity function and, for functions f_i, $i \in \mathbb{N}_{[1,N]}$, $\bigcirc_{i=1}^N f_i$ denotes the composition of functions $f_1 \circ \cdots \circ f_N$.

Next, we define the observed q-variate time series w_d, of length T, as a sequence of values observed over time:

$$w_d = \big(w_d(1), w_d(2), \ldots, w_d(T)\big) \in (\mathbb{R}^q)^T.$$

For a given finite sequence of matrices $A_1, A_2, \ldots, A_i, \ldots, A_r$ with equal number of columns, we define

$$\mathrm{col}(A_1, A_2, \ldots, A_r) := \begin{bmatrix} A_1^\top & A_2^\top & \ldots & A_r^\top \end{bmatrix}^\top. \quad (1)$$

Furthermore, if any element of the matrix sequence, say A_i, is left out, then this case is denoted by $\mathrm{col}(A_1, A_2, \ldots, A_r) \mid A_i$. Finally, we recall the following definition.

Definition 1: [16] A k-variate time series of length T, given by $z := (z(1), z(2), \cdots, z(T))$ is said to be persistently exciting (PE) of order $L \in \mathbb{N}$ if the Hankel matrix constructed with L-block rows as follows:

$$\mathcal{H}_L(z) = \begin{bmatrix} z(1) & z(2) & \cdots & z(T-L+1) \\ z(2) & z(3) & \cdots & z(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ z(L) & z(L+1) & \cdots & z(T) \end{bmatrix}$$

has a rank of kL, indicating full row rank.

III. Problem formulation

We consider a class of control systems, specifically linear time-invariant systems given in discrete-time by:

$$x(t+1) = Ax(t) + Bu(t) := f(x(t), u(t)), \quad t \in \mathbb{N} \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is its control input, $x(0) = x_0$ is its initial state, and the matrices A, B are of appropriate dimensions. For a sequence of control inputs $(u_i)_{i\in\mathbb{N}_{[0,t]}}$, we denote the state of the system at time $t+1$ and whose initial state is x_0 by $x(x_0, (u_i)_{i\in\mathbb{N}_{[0,t]}})$. Linearity of system (2) implies $x(\alpha x_0, \alpha u_0) = \alpha x(x_0, u_0)$ for $\alpha \in \mathbb{R}$. Matrices A, B are unknown; however, we make an experiment collecting input/state measurement data and organize them in the following matrices:

$$U_{0,T} = [u_d(0) \ldots u_d(T-1)], \quad (3a)$$

$$X_{0,T} = [x_d(0) \ldots x_d(T-1)], \quad (3b)$$

$$X_{1,T} = [x_d(1) \ldots x_d(T)]. \quad (3c)$$

Definition 2: LTI system (2) with a given control $u(t) = \phi(x(t))$, $t \in \mathbb{N}$ is *globally exponentially stable* (*GES*) if there exists $\varepsilon \in (0,1)$ and $C \in \mathbb{R}^+$ such that the solution of (2) verify

$$\|x(t)\| \leq C\varepsilon^t \|x(0)\|, \quad t \in \mathbb{N}. \quad (4)$$

To evaluate the performance of the system (2), we further rely on the function:

$$J_L(t) = \sum_{n=1}^{L} q||x_n(t)||_Q + F(x_L(t)), \quad (5)$$

where the design parameter $q \in \mathbb{R}^+$ and the function $F(\cdot)$ are to be defined, $Q > 0$ is a given weighting matrix, and $x_n(t)$ denotes the state $x(t+n)$. In this paper, we consider tackling the following problems:

Problem 1: Given are the LTI system (2), an input/state time series (u_d, x_d), a state constraint set $U \in \mathcal{K}_0(\mathbb{R}^m)$, a control constraint set $U \in \mathcal{K}(\mathbb{R}^m)$, and a cost function $J_L(t)$ defined as (5). We need to synthesize the terminal cost $F(\cdot)$, design parameter q, and the control $u(t) = \phi_L(x(t))$, $t \in \mathbb{N}$, so that the corresponding closed-loop system is *GES*, $u(t) \in U$ for all $t \in \mathbb{N}$, and $J_L(t)$ is minimized at every $t \in \mathbb{N}$.

Problem 2: Given an interconnection $\mathcal{P}$ of $N \in \mathbb{N}^+$ coupled discrete-time linear time-invariant subsystems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$, we need to synthesize controllers for individual subsystems while guaranteeing that the closed-loop interconnected system is *GES*.

In Section IV we solve Problem 1 using our novel approach that is based on data-driven predictive control with terminal constraints using reachable sets. Furthermore, we rigorously define the interconnected system $\mathcal{P}$ and solve Problem 2 using the solution of Problem 1 and small-gain arguments in Section IV-D.

IV. Main results

We review our approach presented in [15] to synthesize, using offline-collected input/state data, a control invariant set for the system (2). This set is essential in designing terminal set constraints and terminal costs in the optimization problem formulated later in the second part of this section.

A. Control invariant set

The following definitions will be used later in synthesizing the control invariant set using backward reachability computations.

Definition 3: A polytope $\mathcal{S} \in \mathcal{K}(\mathbb{R}^n)$ is a closed intersection of a finite set of half-spaces, and is given in:

- half-space representation (H-representation) as

$$S = \{x \in \mathbb{R}^n : Hx \leq 1\}, \tag{6}$$

  where $H \in \mathbb{R}^{l\times n}$,$l \in \mathbb{N}$ and in
- vertex representation (V-representation) as

$$S = \text{ch}(v^{[1]}, \dots, v^{[p]}), \tag{7}$$

  with $\text{vert}(S) = \{v^{[1]}, \dots, v^{[p]}\} \subset \mathbb{R}^n$, and $p \in \mathbb{N}$.

Definition 4: [10] The set $S \in \mathcal{K}_0(\mathbb{R}^n)$ is said to be a control invariant set for LTI system (2) if for every $x \in S$, there exists $u \in U$ such that $f(x,u) \in S$.

We note that if the condition in Definition 4 is satisfied with $f(x,u) \in \rho S$ for a $\rho \in (0,1)$, then the set S is not only an invariant set but also called a contracting set for the closed-loop system.

Definition 5 (N-step reachable set): Given a set $S \in \mathcal{K}_0(\mathbb{R}^n)$ and the set of input constraints U, the N-step reachable set $\mathcal{R}_N(S)$ for $N = 1, 2, \dots$ is recursively defined [10]:

$$\begin{aligned} &\mathcal{R}_j(S) = \{x \in \mathbb{R}^n : \exists u \in U, f(x,u) \in \mathcal{R}_{j-1}(S)\} \cap S \\ &\mathcal{R}_0(S) = S \end{aligned} \tag{8}$$

where $j = 1, \dots, N$.

The sequence $\mathcal{R}_j(S)$ satisfies $\mathcal{R}_{j+1}(S) \subseteq \mathcal{R}_j(S)$ and the infinite-time backward reachability set is defined as:

$$\mathcal{R}_\infty(S) = \bigcap_{i=0}^{\infty} \mathcal{R}_i(S). \tag{9}$$

The following fact follows from [10]:

Corollary 1: If there exists $\bar{i} \in \mathbb{N}$ such that $\mathcal{R}_{\bar{i}+1}(S) = \mathcal{R}_{\bar{i}}(S)$, then $\mathcal{R}_{\bar{i}}(S) = \mathcal{R}_\infty(S)$ and $\mathcal{R}_i(S)$ is invariant for the closed-loop system.

It is necessary in Corollary 1 to know a priori the matrices A and B for the system (2). Nevertheless, in the following we present a result adapted from [15, Proposition 9] that derives the invariant set therein for (2) using offline collected data (u_d, x_d) based on the fundamental lemma [16, Theorem 1].

Lemma 1 (Fundamental lemma): Consider system (2), and let it be controllable. Further, assume that $u_d := \big(u_d(1), u_d(2), \cdots, u_d(T)\big)$ is persistently exciting of order $n+L$. Then $\text{col}(u|_L, x|_L)$ is an L-long input/state trajectory of the system if and only if there exists a $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} u|_L \\ x|_L \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(u_d) \\ \mathcal{H}_L(x_d) \end{bmatrix} g. \tag{10}$$

Proposition 1: Given system (2) and sets S, U having the form in (6) with matrices H_x and H_u, respectively. Then, the N-step reachable set is given in H-representation as

$$\begin{aligned} \mathcal{R}_{j+1}(S) &= \{x \in \mathbb{R}^n : H_{j+1}x \leq 1\} \\ &= X_{0,T}\{g \in \mathbb{R}^T : \begin{bmatrix} H_j X_{1,T} \\ H_u U_{0,T} \\ H_x X_{0,T} \end{bmatrix} g \leq 1\}, \\ H_0 &= H_x, \end{aligned} \tag{11}$$

where $j = 0, \dots, N$.

Using data, and Proposition 1 we compute the N-step reachable set Φ, satisfying the condition

$$\Phi = \mathcal{R}_N(S) = \mathcal{R}_{N+1}(S). \tag{12}$$

Thus using Corollary 1, Φ is a control invariant set for system (2) and we can conclude the first part of this section.

The next part of this section uses the control invariant set Φ in order to reformulate Problem 1 into a moving horizon optimal control problem.

Remark 1: We can construct a contracting set Φ with a contraction $\rho \in (0,1)$ by following the approach in this section while only changing:

$$\mathcal{R}_j(S) = \{x \in \mathbb{R}^n : \exists u \in U, f(x,u) \in \rho\mathcal{R}_{j-1}(S)\} \cap S.$$

Then, (8) and (11) are modified accordingly.

B. Data-driven predictive control with terminal cost

In this section, we provide a solution to Problem 1. We stabilize system (2) while minimizing the cost function $J_T(t)$ in (5) and satisfying input and state constraints. We propose a data-driven predictive control problem for LTI systems of the form (2) at every discrete time $t \in \mathbb{N}$ as the following MPC optimization problem:

$$\begin{aligned} \min_{g,u}\ & J_L(t) \\ \text{s.t.}\ & Eq.\ (10), && \text{(13a)} \\ & u_n(t) \in \mathcal{U}, \quad \forall n \in \mathbb{N}_{[1,L]} && \text{(13b)} \\ & x_n(t) \in \mathcal{X}, \quad \forall n \in \mathbb{N}_{[1,L]} && \text{(13c)} \\ & x_1(t) = x(t), && \text{(13d)} \\ & x_L(t) \in \mathcal{X}_f. && \text{(13e)} \end{aligned}$$

Matrix $Q > 0$ is the state weighting matrices and scalar $q \geq 0$ is a design parameter. Sets $\mathcal{U} \in \mathcal{K}_0(\mathbb{R}^m)$ and $\mathcal{X} \in \mathcal{K}_0(\mathbb{R}^m)$ are assumed to be given closed, bounded, and convex sets. The cost function is given in terms of a terminal cost and the summation of a stage cost

$$J_L(t) = \sum_{n=1}^{L} q\|x_n(t))\|_Q^2 + F(x_L(t)).$$

The objective is thus to track the equilibrium point $x_e = 0$, and minimize the cost related to the terminal state. As for the terminal set $\mathcal{X}_f$ and the terminal cost function $F(\cdot)$, they are design parameters for the controller. Constraint (13a) refers to the equivalent representation of System (2) using previously collected input/state data. (13b)-(13c) define constraints on the inputs and states, (13d) defines the initial state for the system, and (13e) specifies terminal constraints.

The optimal control problem (13) defines a receding horizon predictive controller $u(t) = \phi_L(x(t))$ whose operation is more explained in Section V.

C. Feasibility and stability

In this section we discuss the recursive feasibility and the stability properties of the controller given by (13). Indeed, adding the terminal constraint (13e) and terminal cost $F(x_L(t))$ allows for guaranteeing stability and recursive feasibility. Before stating the result we recall conditions in [18] on $F(\cdot)$ and $\mathcal{X}_f$.

Assumption 1: Given a set $\mathcal{X}_f \subseteq \mathcal{X}$, $\mathcal{U} \subset \mathbb{R}^m$ and a function $F : \mathcal{X}_f \to \mathbb{R}_+$. There exists a controller $\phi : \mathcal{X}_f \to \mathcal{U}$ with the following properties:

- $\mathcal{X}_f$ is invariant for the discrete time system (2),
- function $F(\cdot)$ is a Lyapunov function for system (2) on set $\mathcal{X}_f$ which is compatible with the stage cost in the following sense: for each $x \in \mathcal{X}_f$, the inequality

$$F(f(x, \phi(x))) \leq F(x) - q\|x\|_Q^2 \tag{14}$$

holds.

Theorem 1: Assume that the set $\mathcal{X}_f \in \mathcal{K}_0(\mathbb{R}^n)$ and the terminal cost $F(\cdot)$ in OCP (13) satisfies Assumption 1. Assume further that u_d is persistently exciting of order $n+L$. If the OCP is feasible at $t_0 = 0$ then it is recursively feasible and the closed-loop system is asymptotically stable.

Theorem 1 gives sufficient conditions for stabilizing the system (2). Therefore, solving Problem 1 boils down to synthesizing an appropriate terminal cost and terminal set. We thus state here the main result of this paper as the following:

Theorem 2: Let the set $\mathcal{X}_f$ be a contracting set for the closed-loop system with a contraction $\rho \in (0, 1)$ and a controller $u = \phi_L(\cdot)$. If there exist $\alpha \in \mathbb{R}_0^+$ such that $\rho \leq 1 - \alpha$ and $q\|x\|_Q^2 \leq \alpha F(x)$, $x \in \mathcal{X}_f$, then Assumption 1 is satisfied with $F(x) = \min\{\rho \in \mathbb{R}_0^+ : x \in \rho\mathcal{X}_f\}$.

The existence of a constant α satisfying the conditions of Theorem 2 is not restrictive if the design parameter q in OCP (13) is chosen as stated in the following result.

Corollary 2 (Designing parameters): Let the set $\mathcal{X}_f \in \mathcal{K}_0(\mathbb{R}^n)$ be a contracting set for the closed-loop system withontraction $\rho \in (0, 1)$. Choose the design parameter q in OCP (13) such that $0 \leq q \leq \frac{1-\rho}{\bar{c}^2}$. Then $\alpha = q\bar{c}^2$ satisfies the assumption of Theorem 2, for a scalar $\bar{c}$ satisfying $\mathcal{X}_f \subseteq \bar{c}\mathcal{B}_Q$.

D. Control of interconnected systems

We use the results in the previous sections to analyze, using data, the stability of an interconnection $\mathcal{P}$ of $N \in \mathbb{N}^+$ discrete-time linear time-invariant systems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$. As for the dynamics of $\mathcal{P}_i$, it extends (2) by additional coupling terms:

$$x^{[i]}(t+1) = A_i x^{[i]}(t) + B_i u^{[i]}(t) + \sum_{j=1}^{N} A_{ij} x^{[j]}(t),\ \forall t \in \mathbb{R}_0^+ \tag{15}$$

where $x^{[i]}(t) \in \mathbb{R}^{n_i}$ is the state, $x^{[j]}(t) \in \mathbb{R}^{n_j}$ is the state of system $\mathcal{P}_j$, $j \in \mathbb{N}_{[1,N]}$, $u^{[i]}(t) \in \mathbb{R}^{m_i}$ is the control input, and $A_{ii} = 0$. The state of $\mathcal{P}$ is then $x(t) = [x^{[1]}(t); \ldots; x^{[N]}(t)]$, its control input is $u(t) = [u^{[1]}(t); \ldots;^{[N]}(t)]$, and the initial state is given as $x(0) = [x_0^{[1]}; \ldots; x_0^{[N]}]$. We assume that the control input of system $\mathcal{P}_i$ is designed using the data-driven predictive control approach given in Section IV-B with a prediction horizon $L_i = 2$ and a contracting set $\mathcal{X}_f^{[i]} \subset \mathcal{K}_0(\mathbb{R}^n)$, i.e. $u^{[i]}(t) = \phi_2^{[i]}(x^{[i]}(t))$ for all $t \in \mathbb{R}^+$, $i \in \mathbb{N}_{[1,N]}$. Stability of subsystem $\mathcal{P}_i$ is then studied in the following sense:

Definition 6 (ISS): Given the control input $u^{[i]}(t) = \phi_2^{[i]}(x^{[i]}(t))$, $\mathcal{P}_i$ is input-to-state stable (*ISS*) with respect to $x^{[j]}, j \in \mathbb{N}_{[1,N]} \setminus \{i\}$, if there exist a $\mathcal{KL}$ function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, and $\{\gamma_{i1}, \ldots, \gamma_{iN}\} \in (\mathcal{K} \bigcup \{0\})^{N+1}$ with $\gamma_{ii} = 0$ such that for all $x^{[j]} \in \mathcal{L}_\infty^{n_j}$:

$$||x^{[i]}(t)|| \leq \max\left\{\beta(||x_0^{[i]}||, t), \max_{j \in \mathbb{N}_{[1,N]}} (\gamma_{ij}(||x^{[j]}||))\right\} \tag{16}$$

for all $t \in \mathbb{R}^+$.

Now suppose that each subsystem $\mathcal{P}_i$ is input-to-state stable, with respect to $x^{[j]}, j \in \mathbb{N}_{[1,N]} \setminus \{i\}$. Then, the following lemma gives a sufficient small gain condition (SGC) that guarantees that the interconnection $\mathcal{P}$ is indeed *GES*.

Lemma 2 (SGC [19]): Suppose that each of the subsystems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$, is *ISS*, with respect to $\xi^j, j \in \mathbb{N}_{[1,N]} \setminus \{i\}$, with gains γ_{ij}, $i, j \in \mathbb{N}_{[1,N]}$. If

$$\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \cdots \circ \gamma_{i_{k-1} i_k}(s) < s \tag{17}$$

for all $s \in \mathbb{R}^+$, $i_0, \ldots, i_k \in \mathbb{N}_{[1,N]}$ with $i_0 = i_k$ and $k \in \mathbb{N}_{[1,N]}$, then the interconnected system $\mathcal{P}$ is *GES*.

Lemma 2 suggests that once we have all the *ISS* gains γ_{ij} for systems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$, we can check the stability of the interconnected system $\mathcal{P}$ using the cycle condition (17). Thus, with the controllers designed for each subsystem $\mathcal{P}_i$, solving Problem 2 reduces to analyzing the stability of $\mathcal{P}$, which requires addressing the following gain derivation problem:

Problem 3 (Gain derivation): Consider an interconnection $\mathcal{P}$ of control systems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$. Given are the controllers $\phi_2^{[i]}(\cdot)$, and contracting sets $\mathcal{X}_f^{[i]}$, for $\mathcal{P}_i$ with $i \in \mathbb{N}_{[1,N]}$. Assume that each $\mathcal{P}_i$ is *ISS*, with respect to its inputs $x^{[j]}, j \in \mathbb{N}_{[1,N]} \setminus \{i\}$, with some unknown *ISS* gains γ_{ij}, $j \in \mathbb{N}_{[1,N]}$. Derive γ_{ij}, $i, j \in \mathbb{N}_{[1,N]}$.

As we are interested in solving Problem 3 based on data, we need to recollect and organize data for each system $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$, as the following:

$$U_{0,T}^{[i]} = [\overline{u}_d^{[i]}(0) \ldots \overline{u}_d^{[i]}(T-1)], \tag{18a}$$

$$X_{0,T}^{[j]} = [\overline{x}_d^{[j]}(0) \ldots \overline{x}_d^{[j]}(T-1)], j \neq i,\ j \in \mathbb{N}_{[1,N]} \tag{18b}$$

$$X_{0,T}^{[i]} = [\overline{x}_d^{[i]}(0) \ldots \overline{x}_d^{[i]}(T-1)], \tag{18c}$$

$$X_{1,T}^{[i]} = [\overline{x}_d^{[i]}(1) \ldots \overline{x}_d^{[i]}(T)]. \tag{18d}$$

Based on the above data matrices, system (15) is rewritten as follows

$$X_{1,T}^{[i]} = A_i X_{0,T}^{[i]} + B_i U_{0,T}^{[i]} + \sum_{j=1}^{N} A_{ij} X_{0,T}^{[j]}. \tag{19}$$

The gains γ_{ij} can be expressed in terms of matrices A_{ij} (see Theorem 3). Thus, we want to identify the matrices A_{ij}. It is known [20] that if the matrix $\mathcal{D} := \mathrm{col}(X_{0,T}^{[i]}, U_{0,T}^{[i]}, X_{0,T}^{[J]})$, where $X_{0,T}^{[J]} = \mathrm{col}(X_{0,T}^{[1]}, X_{0,T}^{[2]}, \ldots, X_{0,T}^{[N]}) \mid X_{0,T}^{[i]}$, has full-row rank for all i, then it is possible to identify A_{ij}. Assuming this is the case, we suppose that $\mathcal{V}$ is the Moore-Penrose inverse of the matrix $\mathcal{D}$. Partitioning $\mathcal{V} = \begin{bmatrix}\mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3\end{bmatrix}$ in such a way that it can be multiplied to $\mathcal{D}$ conformably, where $\mathcal{V}_3$ corresponds to the matrix $X_{0,T}^{[J]}$ and it is defined as

$$\mathcal{V}_3 := \begin{bmatrix}\mathcal{V}_{3,i1} & \mathcal{V}_{3,i2} & \ldots, \mathcal{V}_{3,iN}\end{bmatrix}, \tag{20}$$

where $\mathcal{V}_{3,ii}$ matrix is left out. With this, it is straightforward to see that for $j \neq i$, we have

$$A_{ij} = X_{1,T}^{[i]} \mathcal{V}_{3,ij}. \tag{21}$$

Before presenting the main result of this subsection, we state a lemma, borrowed from [21, Lemma 1], and provide a proposition that will be used in proving the main result.

Lemma 3: For any $a, b \in \mathbb{R}^+$ we have

$$a + b \leq \max((\mathcal{I}_d + \lambda)(a), (\mathcal{I}_d + \lambda^{-1})(b)), \tag{22}$$

for any $\lambda \in \mathcal{K}_\infty$.

Proposition 2: Define the maps $\Phi_i : 2^{\mathcal{X}_f^{[i]}} \to 2^{\mathcal{X}_f^{[i]}}$, with $\Phi_i(\mathcal{S}) = \{x' \in \mathcal{X}_f^{[i]} : \exists x \in \mathcal{S}, x' = A_i x + B_i \phi_1^{[i]}(x)\}$, $i \in \mathbb{N}_{[1,N]}$. Then for any $a \in \mathbb{R}^+$, $\mathcal{S} \subseteq \mathcal{X}_f^{[i]}$, such that $a\mathcal{S} \in \mathcal{X}_f^{[i]}$, we have $\Phi_i(a\mathcal{S}) \subseteq a\Phi_i(\mathcal{S})$.

The *ISS* gains for systems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$, are then given using input/state data by the following theorem.

Theorem 3: Consider $N \in \mathbb{N}^+$ and $\mathcal{P}$, an interconnection of systems $\mathcal{P}_i$, $i \in \mathbb{N}_{[1,N]}$. Let $\mathcal{P}_i$ be controlled by $u^{[i]}(t) = \phi_2^{[i]}(x^{[i]}(t))$, using the contracting set $\mathcal{X}_f^{[i]} \subset \mathcal{K}_0(\mathbb{R}^n)$. Then $\mathcal{P}_i$ is *ISS*, with respect to its inputs $x^{[j]}, j \in \mathbb{N}_{[1,N]} \setminus \{i\}$, with gains γ_{ij}, $j \in \mathbb{N}_{[1,N]}$ given by

$$\gamma_{ij}(s) = \frac{1}{1-\rho_i} \frac{\overline{c}_i}{\underline{c}_i} \left(\bigcirc_{l=0}^{j} (\mathcal{I}_d + \lambda_{il}) \right) \circ (\mathcal{I}_d + \lambda_{i(j+1)}^{-1})) || X_{1,T}^{[i]} \mathcal{V}_{3,ij} ||, \tag{23}$$

for constants $\underline{c}_i, \overline{c}_i \in \mathbb{R}^+, \underline{c}_i \leq \overline{c}_i$, $\rho_i \in (0,1)$, $i \in \mathbb{N}_{[1,N]}$, and any functions $\lambda_{ij} \in \mathcal{K}_\infty$, $i \in \mathbb{N}_{[1,N]}$, $j \in \mathbb{N}_{[0,N+1]}$

V. Algorithms for control design

Given now the initial collected data $U_{0,T}, X_{0,T}, X_{1,T}$ as in (3), input constraint set U, state constraint set $\mathcal{X} \in \mathcal{K}_0(\mathbb{R}^n)$, contracting factor $\rho \in (0,1)$, and any set $S \in \mathcal{X}$. We design offline a contracting set $\mathcal{X}_f$ with a contraction ρ, using Remark 1 and following the backward reachable set computation approach in Section IV-A, as described by Algorithm 1. After that we set the parameter q as in Corollary 2, the function $F(\cdot)$ as in Theorem 2 and rewrite OCP (13) as:

Algorithm 1 Contracting set computation

function Cont($U_{0,T}$, $X_{0,T}$, $X_{1,T}$, $\mathcal{U}$, S, ρ)
1: $j = 0$ and $\mathcal{X} := S$.
2: Compute $\mathcal{R}_j(S)$ as in Proposition 1.
3: If $\mathcal{R}_j(S) := \mathcal{X}$, return($\mathrm{X_j}$).
4: Set $\mathcal{X} := \mathcal{R}_j(S)$.
5: Increment the index $j := j + 1$ and return to step 2.

Algorithm 2 DPC with terminal costs

function DPCTC($U_{0,T}$, $X_{0,T}$, $X_{1,T}$, $\mathcal{U}$, $\mathcal{X}$, S, $x(0)$, ρ, q, Q, L)
1: $\mathcal{X}_f :=$ Cont$(U_{0,T}, X_{0,T}, X_{1,T}, U, S, \rho)$.
2: $x(t) := x(0)$.
3: $u(t) := \mathcal{C}(\mathcal{X}_f, x(t), \mathcal{X}, \mathcal{U}, U_{0,T}, X_{0,T}, X_{1,T}, L, q, Q)$.
4: Compute $x(t+1) := f(x(t), u(t))$.
5: Increment the time $t := t + 1$ and return to step 3.

$$\min_{g,u,\rho} \sum_{n=1}^{L} q \| x_n(t)) \|_Q^2 + \rho$$

$$\text{s.t.} \quad Eq.\ (10), \tag{24a}$$

$$u_n(t) \in \mathcal{U}, \qquad \forall n \in \mathbb{N}_{[1,L]} \tag{24b}$$

$$x_n(t) \in \mathcal{X}, \qquad \forall n \in \mathbb{N}_{[1,L]} \tag{24c}$$

$$x_1(t) = x(t), \tag{24d}$$

$$x_L(t) \in \rho \mathcal{X}_f, \tag{24e}$$

$$\rho \leq 1, \tag{24f}$$

OCP (24) is a convex quadratic optimization problem that could be solved efficiently using off the self optimizers like CPLEX, GUROBI [22], or QUADPROG. Then the DPC approach with terminal constraints is summarized by Algorithm 2.

In the first step of Algorithm 2 we compute offline the control invariant set $\mathcal{X}_f$. In steps 2-3, we solve online the optimal control problem (24) for an initial condition $x(0)$ to find the sequence of optimal inputs $(u_n^*(t))_{n \in \mathbb{N}_{[1,L]}}$. The control input at time t is then defined as $u(t) = u_n^*(t)$. OCP (24) is guaranteed to be feasible at each time $t \in \mathbb{N}$ whenever it is feasible for the initial given condition $x(0)$. In line 4 we update the state by feeding back the control input $u(t)$ to the system. Finally, we shift the horizon one step and recompute iteratively the control inputs for every $t\mathbb{N}$.

VI. Illustrative example

We implement the presented algorithms in Matlab using the Multi-Parametric Toolbox [23].

Consider the plant model given by (2) with

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{25}$$

After generating offline data (3) with $T = 150$ by considering persistently exciting input data u_d, we set $\rho = 0.99$. The system is unstable, with all eigenvalues of A ($\lambda_1 = 1.43, \lambda_2 = 5.56$) outside the unit circle in the complex plane, but controllable. We follow the steps of Algorithm 2 and using the backward reachability approach, Algorithm 1, compute a contracting set $\mathcal{X}_f$ for the closed-loop system with a contraction factor ρ and control inputs satisfying $-1 \leq u \leq 1$ and set $S = [-1,1] \times [-1,1]$. We then generate for 10 different initial conditions outside the set $\mathcal{X}_f$ the control inputs using Algorithm 2. For the latter we set the terminal set to be $\mathcal{X}_f, \mathcal{U} = [-1,1]$, $\mathcal{X} = [-1.5, 1.5] \times [-2, 2]$, prediction horizon $L = 10$, $Q = I_2$, and $q = 0.009$. We note that with such a choice of q the conditions of Theorems 1 and 2 are satisfied and thus the closed-loop system is asymptotically stable. Indeed all trajectories converge to the equilibrium while satisfying state constraints as in Figure 1.

Now we consider an interconnection $\mathcal{P}$ of two systems $\mathcal{P}_1$ and $\mathcal{P}_2$ given by (15) with matrices $A_{11} = A_{22} = A$, $B_1 = B_2 = B$, and $A_{12} = A_{21} = 10^{-3} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$. We note that the norm we choose in our computations is based on the set $\mathcal{X}_f$. The set $\mathcal{X}_f \in \mathcal{K}_0(\mathbb{R}^2)$ is symmetric. It follows from Section 3.3 in [10] that the Minkowski functional $||x||_{\mathcal{X}_f} = inf\{\lambda \in \mathbb{R}_0^+, x \in \lambda \mathcal{X}_f\}$ is a norm with unit ball $\mathcal{X}_f$. Then we derive the *ISS* gains γ_{12}, γ_{21} using Theorem 3 and the norm $||\cdot||_{\mathcal{X}_f}$. Next we check the cycle condition (17) to get

$$\gamma_{12}\gamma_{21} \leq 0.9999 < 1. \tag{26}$$

This implies that the overall system $\mathcal{P}$ is globally exponentially stable.

VII. Conclusion

We used data and reachability analysis to design predictive controllers for discrete-time linear systems. After setting the prediction horizon, we use the offline computations of the terminal cost function and terminal constraint to guarantee the recursive feasibility of the online optimization problem and the stability of the closed-loop system. We used the results further to design controllers for interconnected systems.

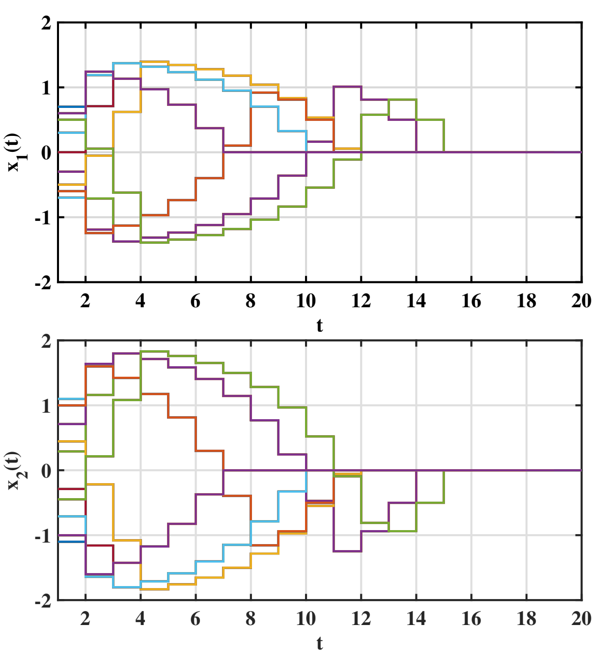

Fig. 1: Trajectories of system (25) generated using the DPC Algorithm 2 with $\mathcal{X}_f$ being the target set.

References

[1] I. Markovsky and F. Dörfler, "Behavioral systems theory in data-driven analysis, signal processing, and control," *Annu. Rev. Control*, vol. 52, pp. 42–64, 2021.

[2] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *Int. J. Control*, vol. 81, no. 12, pp. 1946–1959, 2008.

[3] V. K. Mishra, H. J. van Waarde, and N. Bajcinca, "Data-driven criteria for detectability and observer design for lti systems," in *61st Conf. Decis. Control*. IEEE, 2022, pp. 4846–4852.

[4] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer, "Robust constraint satisfaction in data-driven mpc," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 1260–1267.

[5] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer, "Data-driven model predictive control with stability and robustness guarantees," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1702–1717, 2021.

[6] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer, "On the design of terminal ingredients for data-driven mpc," *IFAC-PapersOnLine*, vol. 54, no. 6, pp. 257–263, 2021.

[7] G. Pan, R. Ou, and T. Faulwasser, "On a stochastic fundamental lemma and its use for data-driven optimal control," *IEEE Transactions on Automatic Control*, pp. 1–16, 2022.

[8] A. Bisoffi, C. De Persis, and P. Tesi, "Controller design for robust invariance from noisy data," *IEEE Trans. Autom. Control*, 2022.

[9] A. Alanwar, A. Koch, F. Allgöwer, and K. H. Johansson, "Data-driven reachability analysis from noisy data," *IEEE Trans. Autom. Control*, 2023.

[10] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[11] P. G. Carlet, A. Favato, S. Bolognani, and F. Dörfler, "Data-driven continuous-set predictive current control for synchronous motor drives," *IEEE Transactions on Power Electronics*, vol. 37, no. 6, pp. 6637–6646, 2022.

[12] A. Alanwar, Y. Stürz, and K. H. Johansson, "Robust data-driven predictive control using reachability analysis," *European Journal of Control*, vol. 68, p. 100666, 2022.

[13] S. K. Mulagaleti, A. Bemporad, and M. Zanon, "Data-driven synthesis of robust invariant sets and controllers," *IEEE Control Systems Letters*, vol. 6, pp. 1676–1681, 2021.

[14] A. Bisoffi, C. De Persis, and P. Tesi, "Data-based guarantees of set invariance properties," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 3953–3958, 2020.

[15] M. Al Khatib, V. K. Mishra, and N. Bajcinca, "Data-driven control for linear systems using reachable sets," in *2023 European Control Conference (ECC)*. IEEE, 2023, pp. 1–6.

[16] J. C. Willems, P. Rapisarda, I. Markovsky, and B. De Moor, "A note on persistency of excitation," *Syst. Control Lett.*, vol. 54, no. 4, pp. 325–329, 2005.

[17] M. Al Khatib, V. K. Mishra, and N. Bajcinca, "Data-driven predictive control for interconnected systems using terminal ingredients and reachable sets (full version)," 2025. [Online]. Available: https://shorturl.at/3XzNb

[18] L. Grüne, J. Pannek, L. Grüne, and J. Pannek, *Nonlinear model predictive control*. Springer, 2017.

[19] S. Dashkovskiy, B. Rüffer, and F. Wirth, "An iss small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.

[20] H. J. Van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control," *IEEE Trans. Autom. Control*, vol. 65, no. 11, pp. 4753–4768, 2020.

[21] A. Swikir and M. Zamani, "Compositional synthesis of finite abstractions for networks of systems: A small-gain approach," *Automatica*, vol. 107, pp. 551–561, 2019.

[22] B. Bixby, "The gurobi optimizer," *Transp. Res. Part B*, vol. 41, no. 2, pp. 159–178, 2007.

[23] M. Kvasnica, P. Grieder, M. Baotić, and M. Morari, "Multi-parametric toolbox (mpt)," in *International workshop on hybrid systems: Computation and control*. Springer, 2004, pp. 448–462.