Physics-Informed State Observer for Unknown Linear Autonomous Systems with Noisy Measurements

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***Abstract*— State estimation is a pivotal element in navigation tasks of autonomous vehicles. This technique is mainly applied when either a required measurement is not available or when the amount of available sensors in the platform are limited. Most of the state estimation algorithms available in on-board control modules use kinematic models as prior model to estimate the states of the autonomous system. However, these simple kinematic models do not consider dynamic terms and physical properties which can lead to biased state estimates. To overcome this issue, this paper proposes a physics-informed state observer for unknown linear systems under partial and noisy measurements. The proposed approach fuses two complementary concepts for state estimation and dynamics identification. The proposed approach is capable to obtain reliable state estimates whilst attenuating the level of noise. Lyapunov stability is used to derive an appropriate update law for the construction of physics-informed estimate model. Simulation studies are given to show the advantages and challenges of the proposed approach.**

I. INTRODUCTION

Recent applications involving autonomous systems require access to signals that help them to recognize better the surrounding environment [1]. This require the use of state estimation algorithms as well as sensor fusion techniques [2]. State estimation techniques aim to estimate unmeasurable states from the states measurements provided by the available sensors in the autonomous platform [3]–[5]. State observers [6] are the most popular techniques used for state estimation. Particularly, the Luenberger observer [7] has been widely used for state estimation and combined with control techniques such as the linear quadratic regulator (LQR) [8]. The main assumption of traditional state observes is complete knowledge of the system dynamics. Here, modelling error or noisy measurements prevent the algorithm to obtain reliable state estimates which can cause threats in real autonomous system applications.

The optimal state observer known as Kalman filter [9] aims to overcome the modelling and measurement errors in standard state observers. The key idea is to use a probabilistic approach that assumes a Gaussian distributed error and noise in both the system dynamics and output measurements [10], [11]. However, its main drawback appears when the prior motion model is inaccurate which can lead to divergence of the algorithm.

In the last decade, neural networks (NNs) have been used as alternative model-free methods [12], [13] to overcome the issues of state observers and Kalman filters. In this case NNs work as a regression model of the autonomous platform dynamics using either direct-solution or time-steppers models [14]. In the one hand, direct-solution models [15] estimate the state trajectories given some inputs and initial value. On the other hand, stepper models [16] are based on neural solvers [17], [18] for a continuum prediction of the dynamics. These approaches are mainly designed when the full state is available. Recurrent neural networks (RNNs) [19] and reservoir computing approaches [20] have been used as an alternative network to capture time dependencies from the data accurately. KalmanNet [21], [22] has been developed to combine both benefits of Kalman filter and neural networks by providing adaptation of the covariance matrix of both the Gaussian distributed modelling error and sensor noise. This approach is data hungry such that its performance can be poor if the data is not rich. Physics-informed (PI) architectures has gained attraction in the last decade to improve the prediction precision of NN models, whilst giving a physical meaning to the learned model [23]. PI models are usually implemented in the loss function as a regularisation term. However, this requires knowledge of the physical model which can be prohibitive in some application involving autonomous systems.

One tool that has been used to avoid knowledge of prior model is reinforcement learning [24] (RL). Here, the dynamics of the model is inferred from the data by solving in a recursive mechanism a Hamilton-Jacobi-Bellman (HJB) equation [25]. The inference of the dynamics is ensured by fulfilling a persistent of excitation condition (PE) also known as Uniform Complete Observability (UCO) [26]. This is an interesting property that can be exploited for state estimation algorithms. However, the control nature of RL models have limited their usage to control tasks without considering potential state estimation applications.

In view of the above, we are inspired in (i) the model uncertainty challenge, (ii) the learning regularisation of NNs via PI models, and (iii) the model-free nature of RL algorithms. So, this paper proposes a physics-informed state observer that deal with the model-uncertainty challenge using a model-free physics-informed architecture. The approach uses a parameterization of the dynamics of the autonomous systems as a physics informed (PI) model to estimate the unmeasurable states. The weights of the PI model are updated from the error between the real and PI measurement vectors. Stability of the PI observer is verified using Lyapunov stability theory. Simulation studies using a F-16 aircraft and quadcopter models are conducted to demonstrate the benefits of the proposed methodology, challenges and future work.

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The main contributions of this work are twofold:

1) A novel physics-informed based state observer is proposed that allows the accurate estimation of the system states under partial observability and noise presence.
2) The proposed approach allows to estimate both the estimates and parameters of the model simultaneously, combining the inference capabilities of state observers and system identification algorithms.

II. Preliminaries

Consider an autonomous system described by the following linear time-invariant (LTI) model [6]

$$\begin{aligned} \dot{x} &= Fx \\ y &= Hx, \quad x(0) = x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ describes the state vector, $y \in \mathbb{R}^p$ defines the sensor measurements, $F \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{p\times n}$ are the system and output matrices, respectively. The system matrix F is assumed unknown.

Assumption 1: The matrix F is Hurwitz, such that the dynamics of (1) is asymptotic stable [27]. The pair (F, H) is assumed observable.

A simple Luenberger state observer [28] is considered to express the stable dynamics in terms of the available measurements

$$\begin{aligned} \dot{x} &= Fx + K(y - y) \\ y &= Hx, \end{aligned} \tag{2}$$

where $K \in \mathbb{R}^{n\times p}$ is the observer gain. The LTI model (1) can be simplified into

$$\dot{x} = (F + KH)x - Ky. \tag{3}$$

Assumption 2: Matrix K is chosen such that $F+KH < 0$ is a negative definite matrix..

The system (3) can be written in the Laplace domain as

$$X(s) = (sI - \mathcal{K})^{-1}x_0 - (sI - \mathcal{K})^{-1}KY(s), \tag{4}$$

where $\mathcal{K} = F + KH \in \mathbb{R}^{n\times n}$. The time domain solution of (3) is

$$x(t) = e^{\mathcal{K}t}x_0 - \int_0^t e^{\mathcal{K}(t-\tau)}Ky(\tau)d\tau. \tag{5}$$

From (5) we can observe a direct relation between the state and the output measurements [29]. This can help us to obtain a simple dynamic structure for the proposed state estimation algorithm. We can rewrite (4) as a sum of the contributions of each output signal.

$$X(s) = (sI - \mathcal{K})^{-1}x_0 - (sI - \mathcal{K})^{-1}\sum_{i=1}^{p} K^iY^i(s), \tag{6}$$

where K^i is the ith column of matrix K; analogously, the term Y^i is the ith term of the vector Y. The characteristic polynomial of $\mathcal{K}$ is

$$P(s) = \det(sI - \mathcal{K}) = s^n + \lambda_{n-1}s^{n-1} + \cdots + \lambda_1 s + \lambda_0,$$

with $\lambda_i > 0$. Then,

$$\sum_{i=1}^{p}(sI - \mathcal{K})^{-1}K^iY^i(s) := \sum_{i=1}^{p}\frac{\text{adj}(sI - \mathcal{K})}{P(s)}K^iY^i(s), \tag{7}$$

where adj$(\cdot)$ is the adjoint matrix. We can split the numerator of (7) as

$$\begin{aligned} \text{adj}(sI - \mathcal{K})K^i &= \begin{bmatrix} \beta^{i1}_{n-1}s^{n-1} + \cdots + \beta^{i1}_1 s + \beta^{i1}_0 \\ \beta^{i2}_{n-1}s^{n-1} + \cdots + \beta^{i2}_1 s + \beta^{i2}_0 \\ \vdots \\ \beta^{in}_{n-1}s^{n-1} + \cdots + \beta^{in}_1 s + \beta^{in}_0 \end{bmatrix} \\ &= \begin{bmatrix} \beta^{il}_0 & \beta^{il}_1 & \cdots & \beta^{il}_{n-1} \\ \beta^{i2}_0 & \beta^{i2}_1 & \cdots & \beta^{i2}_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta^{in}_0 & \beta^{in}_1 & \cdots & \beta^{in}_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \\ &= W^iN(s) \end{aligned}$$

where $W^i \in \mathbb{R}^{n\times n}$ stores each coefficient of the adjoint matrix and $N(s) \in \mathbb{R}^n$ is a n-dimensional vector of powers of s. Then,

$$\sum_{i=1}^{p}\frac{\text{adj}(sI - \mathcal{K})}{P(s)}K^iY^i(s) = \sum_{i=1}^{p}W^i\frac{N(s)}{P(s)}Y^i(s). \tag{8}$$

Here, the polynomials in $N(s)/P(s)$ introduce n filters that are applied into the measurements vector y. Each filter can be modelled by the following stable linear system

$$\dot{\chi}^i = A\chi^i + By^i, \quad \chi^i(0) = 0. \tag{9}$$

where χ^i is the state of the ith filter and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\lambda_0 & -\lambda_1 & -\lambda_2 & \cdots & -\lambda_n \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Notice that $\det(sI - A) = P(s)$. So, the time-domain version of (7) is

$$\mathcal{L}^{-1}\left\{\sum_{i=1}^{p}\frac{\text{adj}(sI - \mathcal{K})}{P(s)}K^iY^i(s)\right\} = Q\chi, \tag{10}$$

with $Q = [Q^1, Q^2, \cdots, Q^p] \in \mathbb{R}^{n\times np}$ and $\chi = [(\chi^1)^\top, (\chi^2)^\top, \cdots, (\chi^p)^\top]^\top \in \mathbb{R}^{np}$.

So (5) is rewritten as

$$x(t) = e^{\mathcal{K}t}x_0 - Q\chi. \tag{11}$$

The term $e^{\mathcal{K}t}x_0$ converges to zero as $t \to \infty$ with an exponential decay. Thus, without loss of generality we can set this term into zero or assume that $x_0 = 0$. The dynamics of (11) is given by

$$\begin{aligned} \dot{x} &= -Q\dot{\chi} \\ &= -Q\begin{bmatrix} A & \cdots & A \end{bmatrix}\chi - Q\begin{bmatrix} B & \cdots & B \end{bmatrix}y \\ &= -Q(A_y\chi + B_y y) \\ &= -\Psi^\top W, \end{aligned} \tag{12}$$

where A_y and B_y are block diagonal matrices given by $A_y = \mathrm{diag}\{A, \cdots, A\} \in \mathbb{R}^{np\times np}$, and $B_y = \mathrm{diag}\{B, \cdots, B\} \in \mathbb{R}^{np\times n}$. The matrix $\Psi = \Psi(y, \chi) \in \mathbb{R}^{r\times n}$ defines a matrix of basis functions and $W \in \mathbb{R}^r$ stands to the vector of weights given by

$$\Psi = I_n \otimes (A_y\chi + B_y y), \quad W = \mathrm{vec}(Q). \tag{13}$$

Notice that $QB_y = K$. In the next section the physics-informed (PI) state observer is designed based on the measurements vector y.

III. Physics-Informed State Observer

Fig. 1 shows the high-level diagram of the proposed approach. A subsystem composed of p linear systems $\dot{\chi}^i$ is constructed from the p outputs. The outputs of these models are used to construct a estimated model of the autonomous system dynamics. This model acts as a physics-informed regulariser of the overall state estimation algorithm. The error e between each measurement vector feed a gradient rule to update the weights of the PI model. This in consequence stabilizes the state estimations, whilst reducing the level of noise.

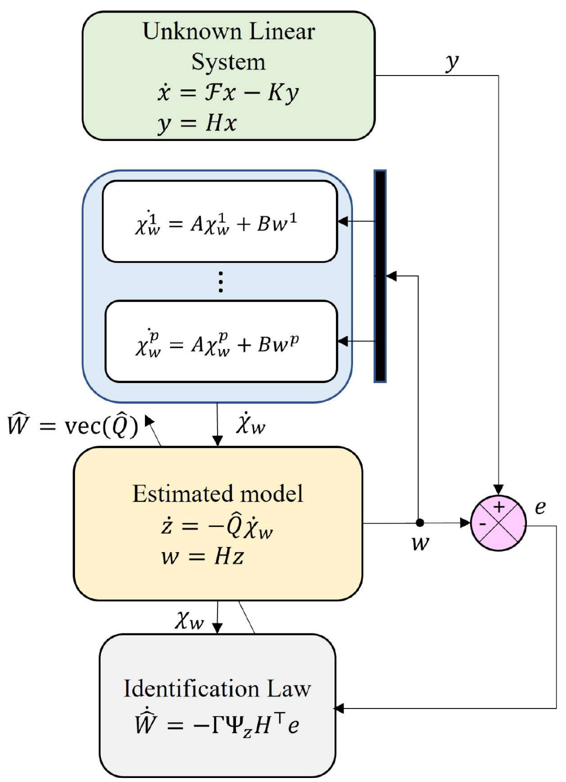


Fig. 1. High-level diagram of the PI state observer architecture

The PI model has the same structure as in (3) as

$$\begin{aligned} \dot{z} &= \widehat{\mathcal{K}}z - \widehat{K}w \\ w &= Hz, \quad z(0) = z_0 = x_0, \end{aligned} \tag{14}$$

where $z \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^p$ is the measurements vector, $\widehat{\mathcal{K}}$ and $\widehat{K}$ are estimates of $\mathcal{K}$ and K. The PI model (14) can be written as in (12) as

$$\begin{aligned} \dot{z} =& \quad -\widehat{Q}(A_y\chi_w + B_y w) \\ =& \quad -\Psi_z^\top \widehat{W}, \end{aligned} \tag{15}$$

where $\widehat{Q} \in \mathbb{R}^{n\times np}$ is the estimate of Q, $\Psi_z = \Psi_z(w, \chi_w) \in \mathbb{R}^{r\times n}$ is the matrix of basis functions based on the output vector of the PI model and $\widehat{W} \in \mathbb{R}^r$ is the vector of weight estimates, which are given by

$$\begin{aligned} \Psi_z &= I_n \otimes (A_y\chi_w + B_y w), \\ \widehat{W} &= \mathrm{vec}(\widehat{Q}). \end{aligned} \tag{16}$$

The state vector $\chi_w = [(\chi_w^1)^\top, (\chi_w^2)^\top, \cdots, (\chi_w^n)^\top]^\top \in \mathbb{R}^{np}$ is computed using the following i linear filters

$$\dot{\chi}_w^i = A\chi_w^i + Bw^i. \tag{17}$$

The output error [30], [31] between the measurements vectors y and w is given by $e = y - w$. Then, the closed loop system between (12) and (15) is

$$\begin{aligned} \dot{e} =& H\Big(- Q(A_y\chi + B_y y) + \widehat{Q}(A_y\chi_w + B_y w)\Big) \\ =& - H\Big(QA_y(\chi - \chi_w) + QB_y e - \widetilde{Q}(A_y\chi_w + B_y w)\Big) \\ =& - H\Big(QA_y(\chi - \chi_w) + QB_y e - \Psi_z^\top\widetilde{W}\Big) \\ =& - H(QB_y e - \Psi_z^\top\widetilde{W} + \varepsilon) \end{aligned} \tag{18}$$

where $\widetilde{Q} = \widehat{Q} - Q \in \mathbb{R}^{n\times np}$ defines the weights error, $\widetilde{W} = \mathrm{vec}(\widetilde{Q}) \in \mathbb{R}^r$, and $\varepsilon = QA_y(\chi - \chi_w)$ is a residual error which is bounded as $\|\varepsilon\| \leq \bar{\varepsilon} \geq 0$. It is assumed that the matrix of basis functions Ψ_z satisfies a persistent of excitation (PE) condition [32], [33].

The uniform ultimate boundedness (UUB) of (18) is discussed in the next theorem.

Theorem 1: Consider the closed-loop output error dynamics (18) with a basis function matrix Ψ_z that is PE. Define $\delta = \lambda_{\min}(HK)$ which satisfies the following inequality

$$\delta > \lambda_{\max}(H)\bar{\varepsilon} + \varrho \tag{19}$$

where $\varrho \in \mathbb{R}^+$. The trajectories of (18) are UUB with a practical bound $\nu = \frac{\lambda_{\max}(H)}{\delta}\bar{\varepsilon}$ if the estimates $\widehat{W}$ are updated using

$$\dot{\widehat{W}} = \dot{\widetilde{W}} = -\Gamma\Psi_z H^\top e, \tag{20}$$

where $\Gamma \in \mathbb{R}^{r\times r}$ is a positive diagonal matrix. This in consequence means that $\widehat{W}$ remains bounded.

Proof: The following Lyapunov function is used

$$V = \frac{1}{2}\|e\|^2 + \frac{1}{2}\widetilde{W}^\top\Gamma^{-1}\widetilde{W} \tag{21}$$

The derivative of (21) along the closed-loop trajectories (18) with the update rule (20) is

$$\begin{aligned} \dot{V} =& - e^\top H(QB_y e - \Psi_z^\top\widetilde{W} + \varepsilon) + \widetilde{W}^\top\Gamma^{-1}\dot{\widetilde{W}} \\ =& - e^\top HQB_y e + \widetilde{W}^\top(\Psi_z H^\top e + \Gamma^{-1}\dot{\widetilde{W}}) - e^\top H\varepsilon \\ =& - e^\top H(QB_y e + \varepsilon) \\ \leq& - \lambda_{\min}(HK)\|e\|^2 + \lambda_{\max}(H)\|\varepsilon\|\|e\| \\ =& - \delta\|e\|\left(\|e\| - \frac{\lambda_{\max}(H)\bar{\varepsilon}}{\delta}\right). \end{aligned}$$

Therefore, $\dot{V}$ is negative definite if

$$\|e\| > \frac{\lambda_{\max}(H)}{\delta}\bar{\varepsilon} \equiv \nu \tag{22}$$

If the gain K verifies (19), then the trajectories of (18) converge to a small set S_ν of radius ν, i.e., $\|e\| \leq \nu$ and therefore, the closed-loop trajectories of (18) are UUB.

To show that $\widehat{W}$ is bounded we rewrite (20) as the following time varying system [34]

$$\begin{aligned} \dot{\widehat{W}} &= \Gamma \Psi_z H^\top u_y \\ \sigma &= \Psi_z^\top \widehat{W} \end{aligned} \tag{23}$$

where $\sigma \in \mathbb{R}^n$ is the measurement vector, $u_y = e$ is a virtual control, and $B(t) = -\Gamma \Psi_z H^\top \in \mathbb{R}^{r\times n}$. The boundedness of e implies boundedness of w. This in consequence means that the basis function matrix Ψ_z is as well bounded. Then, the measurement vector

$$\sigma \equiv -HQB_y e - \dot{e} - \varepsilon. \tag{24}$$

is also bounded. Since both $u_y = e$ and σ are bounded, and Ψ_z is PE, then from the UCO result [34] yields that the weights error $\widetilde{W}$ is bounded, which means that $\widehat{W}$ is bounded as well. This completes the proof. ■

The matrix gain Γ is manually tuned to ensure parameter estimates convergence for each parameter within the parametrized physics-informed model.

IV. Simulation Results

The effectiveness of the PI state observer is demonstrated using different air platforms.

A. F-16 aircraft model

A F-16 aircraft model [35] which fulfils the following linear model

$$\dot{x} = \begin{bmatrix} -1.019 & 0.905 & -0.002 \\ 0.822 & -1.077 & -0.175 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = Hx, \quad H = [I_2, 0_{2\times 1}].$$

We add a Gaussian distributed random noise into the the measurements vector to model sensor noise.

The aircraft is stabilised by any unknown stabilizing control input of the form $u = -Lx$ where $L \in \mathbb{R}^{1\times 3}$ is the control gain. Since L is unknown and stabilise the aircraft model, then we can set any stable characteristic polynomial for the proposed design. We set the system's poles at $-7, -6, -5$, that is, $P(s) = (s+7)(s+6)(s+5)$.

The PI model (15) of the aircraft model is

$$\dot{z} = \exp^{\mathcal{K}t} z_0 - \widehat{W}(A_y \chi_w + B_y w),$$

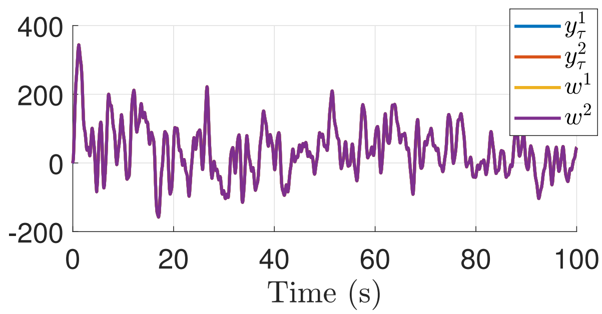
where $\widehat{W} \in \mathbb{R}^{3\times 6}$. So we need to estimate 18 weights. The gain of the update law was heuristically tuned through experimentation. We use the following update gain $\Gamma = 1000 I_{18}$. The term associated to the initial condition is approximated using the eigendecomposition

$$e^{\mathcal{K}t} x_0 = P e^{\mathcal{D}t} P^{-1} x_0$$

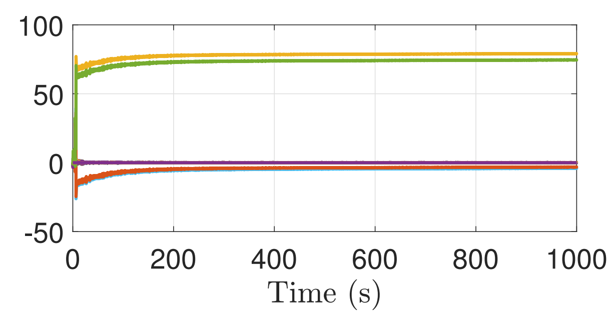
where $x_0 = z_0 = [1, -1, 1]^\top$, $\mathcal{D} = -\mathrm{diag}\{7, 6, 5\}$, and

$$P = \begin{bmatrix} -0.0392 & 0.0274 & -0.0202 \\ 0.1960 & -0.1643 & 0.1414 \\ -0.9798 & 0.9860 & -0.9897 \end{bmatrix}.$$

Fig. 2(a) shows the state estimation result which exhibit high accuracy with noise attenuation capabilities. The convergence of the weights estimates $\widehat{W}$ is shown in Fig. 2(b). Here, the use of the PI model helps to maintain good state estimates throughout the length of the simulation without divergence or instability issues.



(a) Measurements vector comparisons



(b) Weight estimates $\widehat{W}$ convergence

Fig. 2. State estimation results

Fig. 3 shows the estimated state using the proposed approach. Notice that high accurate results are obtained despite the of the lack of the dynamics of the aircraft and a sensor measurement.

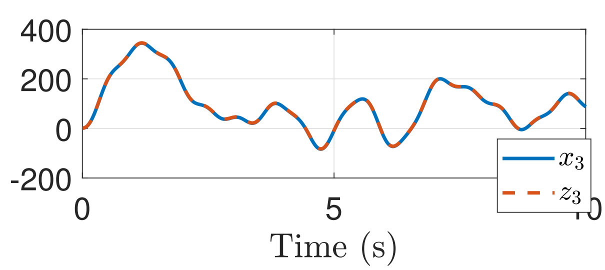


Fig. 3. State estimation of x_3

The error between the measurements vector is equivalent to the added noise. This means that the PI model recovers the real states with noise suppression.

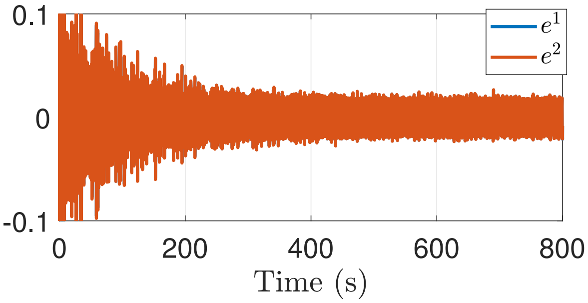


Fig. 4. Output error trajectories

B. Quadcopter model

A quadcopter linearised model [36] in the hover condition is considered

$$\dot{x} = \begin{bmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{bmatrix} x + \begin{bmatrix} 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 1} \\ g & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & 1/m \end{bmatrix} u, \tag{25}$$

where $x \in \mathbb{R}^6$ is the vector of linear positions and velocities, $u \in \mathbb{R}^4$ is the control input composed by the Euler angles and the total thrust, $g = 9.81\text{m/s}^2$ is the gravitational acceleration and $m = 0.467$ is the quadcopter mass. We set the poles of the observer at $\lambda_{1,2} = -5.0184 \pm 0.0106j$, $\lambda_{3,4} = -5 \pm 0.0212j$, and $\lambda_{5,6} = -4.9817 \pm 0.0106j$. We used an update gain of $\Gamma = 500I_{108}$. The weight estimates convergence is shown in Fig. 5.

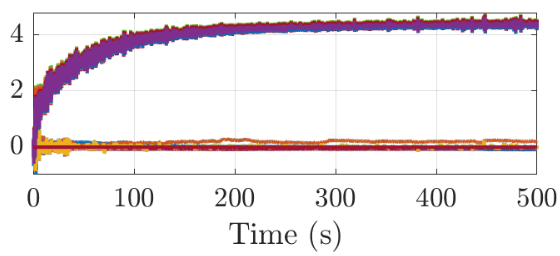


Fig. 5. Weight estimates $\widehat{W}$ convergence

The results show that the weights converge as time increases. The output error results are shown in Fig. 6.

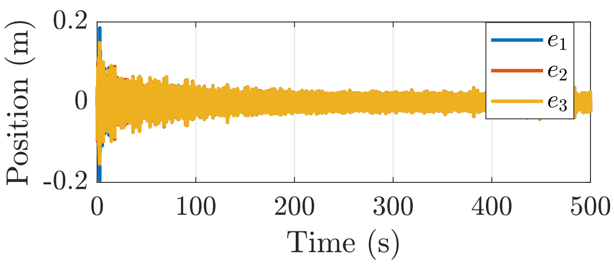


Fig. 6. Output error trajectories

Notice that the output error returns just the added noise in the output measurements, which means the estimated states approximate to the real state estimates. Fig. 7 demonstrates this fact by obtaining high accurate state measurements without noise.

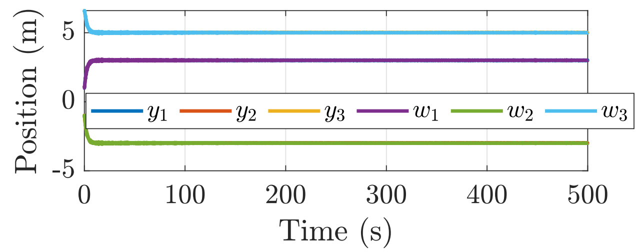


Fig. 7. State estimates with noise attenuation capabilities

V. Conclusions

This paper proposes a novel physics-informed based state observer for unknown linear systems under partial and noisy states measurements. The approach consists of a data-driven algorithm that use a parameterization of the solution of the dynamics of the autonomous system in terms of p linear filters of the output vector. A physics-informed estimated model based on the structure of the parametrized model is constructed to estimate the unobservable states, whilst it attenuate the level of noise. The stability of the PI observer is verified using Lyapunov stability theory. Simulation studies using a F-16 aircraft and quadcopter models are given as case studies to show the effectiveness of the proposed method with accurate state estimation results with noise attenuation capabilities. Future work will study how the proposed work can be applied for a class of Lipschitz nonlinear systems under partial states measurements and exogenous disturbances.

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