Distributed Novelty-Biased Cooperative Protocol

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***Abstract*— We deal with the problem of incorporating the most recent information in the system which may be crucial to update the system state, while maintaining resilience to faulty measurements. For this purpose, we propose and analyze a consensus protocol biased towards the latest information in the system. The convergence to consensus of the system is analyzed and two main approaches are introduced with stability ranges provided from sufficient conditions. A detailed analysis of weight-changing that achieves bias towards novelty in the system is provided. Numerical simulations demonstrate that final value and convergence rate of the system can be controlled. Finally, we provide system demonstration in a simple use-case example.**

I. Introduction

Multi-agent system consists of a set of agents that interact within a shared environment, where each agent typically has only partial knowledge of that environment. To accomplish a task, agents share their information and form a global view of the world [1]. One of the commonly used concepts to merge distributed information is consensus [2], an inherently distributed method that allows agents to agree upon a certain value in the system. One of the simplest consensus types is the average consensus [2], where the agents' values converge to the average of their initial values. Another common type, where final value can be biased according to some criterion, is the weighted consensus [2], [3], [4]. Information of other agents can contribute to the agreed value with a greater or lesser weight. The criterion on choosing these weights can be i.e. how much a certain agent is trusted as reliable [5], the quality of the connection link in a networked systems [6], the uncertainty of measurements [7], etc.

A common scenario is where the agents are exploring the environment to locate the target wherefore they need to make decisions that rely on the history of sensor readings, which have some uncertainty due to noisy measurements. One such common decision-making framework is Partially Observable Markov Decision Processes (POMDPs), where agent builds a belief about the state of the environment based on the sensor readings. In a system involving multiple agents, efficiency can be significantly improved if agents share their knowledge with their peers. This is where the consensus becomes useful, allowing agents to share knowledge, agree on a certain value by filtering the information. The traditional average consensus [2] treats all information as equally important, no matter when and from which agent it originates. That can be a problem from the point of view of an agent introducing new information into the system. Due to the averaging in the consensus, this new information is considered to be of equal importance as existing (old) information, which can cause degradation in the overall belief update of the system. To prevent this we introduced weight-changing consensus that prioritizes the recent information entering in the system [8], called *novelty-biased* (NB) *consensus*. In this work we provide formal analysis of its convergence to consensus and we also explore two distinct methods to introduce bias: node-based and edge-based weight-changes [9]. In the node-based approach, the agent that obtains new information places less emphasis on the information from other agents. In the edge-based approach, other agents assign greater weight to the most recent information introduced into the system. While our method belongs to the class of weight-changing consensus techniques commonly discussed in the literature, we preserve the underlying graph structure and modify only the gains associated with information exchange. Therefore, we refer to our approach as gain-changing consensus. To make this consensus directly applicable in the real world scenarios, we define it in discrete-time domain and set the conditions on the time step maintaining the stability and the bias desired in the system, under assumption that the communication graph is bidirectional and connected. To prove the stability we use Lyapunov based method, which we decided to use to obtain the conditions to showcase the formal reasoning on the monotonicity of weighting functions in two different cases and additionally to easily upgrade the future work on the convergence rates. The main contributions of the paper are as follows:

- Formalization of novelty-biased consensus [8] through the concept of weight-changing consensus,
- Analysis of the convergence to consensus of the proposed weight-changing algorithm,
- Two different ways of changing the weights that ensure convergence to consensus and introduce bias towards the novel information in the system.

The paper is structured as follows. In Section II preliminaries on the graph theory and consensus are given after which the gain-changing consensus is introduced in Section III and its convergence to final value in Section IV. Stability analysis of two types of gain-changing consensus is presented in V. Section VI introduces two gain-changing functions for novelty-biased consensus. An example of a

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simple use-case is presented in VII. We conclude the paper in Section VIII.

II. Preliminaries

Consider a set of n agents whose communication and information flow are represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the set of nodes $\mathcal{V} = \{1, 2, \cdots, n\}$ represents agents and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} = \{e_{ij}\}, i, j \in \mathcal{V}$ corresponds to communication links between agents. The neighbourhood of agent i is a set of all the agents that have a communication link with an agent i, $\mathcal{N}_i = \{j : e_{ij} \in \mathcal{E}\}$. Graph is mathematically described by its adjacency ($\mathbf{A}$) matrix. $\mathbf{A} = \{a_{ij}\}, \mathbf{A} \in \mathbf{R}^{n\times n}$, a square binary matrix where $a_{ij} > 0$, if $e_{ij} \in \mathcal{E}$, otherwise $a_{ij} = 0$. For undirected graphs, $e_{ij} \in \mathcal{E} \Leftrightarrow e_{ji} \in \mathcal{E}$, $\mathbf{A}$ is symmetric. Sum of a row i in matrix $\mathbf{A}$ represents the in-degree (d_i) of an agent i, which forms a diagonal degree matrix $\mathbf{D} = \mathrm{diag}([d_1, d_2, \cdots, d_n])$, where $\mathbf{D} \in \mathbb{R}^{n\times n}$. The difference of degree and adjacency matrix is the Laplacian matrix of a graph ($\mathbf{L} = \mathbf{D} - \mathbf{A}$), with the property of zero row sums. A graph is said to be connected if there is a path between every pair of its vertices.

A. Consensus Protocol

In consensus protocol agents communicate their state $x_i(k)$ over the previously described graph $\mathcal{G}$ and calculate new state value $x_i(k+1)$ based on the received states $(x_j(k), j \in \mathcal{N}_i)$ as:

$$x_i(k+1) = x_i(k) + t_s \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)). \quad (1)$$

This is the discrete time average consensus protocol where t_s is the sampling time. To introduce a bias in the system the weight parameter $\omega_{ij} \in \mathbb{R}_{\geq 0}$ is defined. That kind of a system can be represented with three weighted matrices: weighted adjacency matrix $\mathbf{A}_w$, weighted degree matrix $\mathbf{D}_w$ and weighted Laplacian matrix $\mathbf{L}_w$. Weighted degree matrix, $\mathbf{D}_w$ is defined as:

$$\mathbf{D}_w = \mathrm{diag}([d_{11}, \cdots, d_{nn}]), d_{ii} = \sum_{j\in\mathcal{N}_i} \omega_{ij}, \quad (2)$$

where ω_{ij} presents edge weight. Weighted adjacency matrix $\mathbf{A}_w$ has elements: $\omega_{ij} = a_{ij}\gamma_{ij}$, where a_{ij} is the element of the constant binary adjacency matrix $\mathbf{A}$ and $\gamma_{ij} > 0$ is the element of the gain matrix, which is based on how the agent i should treat the information from the agent j without changing the underlying graph structure. General gain matrix ($\mathbf{\Gamma}$) is defined as:

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{bmatrix}. \quad (3)$$

The element of the weighted Laplacian matrix $\mathbf{L}_w = \mathbf{D}_w - \mathbf{A}_w$, is presented as:

$$l_{ij} = \begin{cases} -a_{ij}\gamma_{ij} & , \ i \neq j \text{ and } j \in \mathcal{N}_i \\ \sum_{j\in\mathcal{N}_i} a_{ij}\gamma_{ij} & , \ i = j \end{cases}. \quad (4)$$

The weighted consensus thus can be written as:

$$x_i(k+1) = x_i(k) + t_s \sum_{j\in\mathcal{N}_i} \omega_{ij}(x_j(k) - x_i(k)). \quad (5)$$

Lemma II.1. *[10] Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w$ (4). This Laplacian matrix has:*

- *at least one eigenvalue equal to 0*
- *all other eigenvalues with non-negative real parts, which makes it a positive semi-definite matrix*
- *zero-row sum, i.e. $\mathbf{L}_w\mathbf{1}_n = \mathbf{0}_n$, meaning that the right eigenvector associated with eigenvalue 0 (right-zero eigenvector) is a column vector of ones $\mathbf{1}_n$*
- *left eigenvector associated with eigenvalue 0 (left-zero eigenvector) is $\mathbf{p}_w \in \mathbb{R}^n$, where ($\mathbf{p}_w^T\mathbf{L}_w = \mathbf{0}_n$)*

The significance of the left-zero eigenvector is crucial because the final consensus value depends on it.

Lemma II.2. *[11] Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w$ (4) with left-zero eigenvector $\mathbf{p}_w$. Final consensus value is then:*

$$x^* = \frac{\mathbf{p}_w{}^T\mathbf{x}_0}{\mathbf{p}_w{}^T\mathbf{1_n}}, \quad (6)$$

where $\mathbf{x}_0 \in \mathbb{R}^n$ is the vector of initial conditions of the state vector $\mathbf{x}(k) \in \mathbb{R}^n$.

The pertaining synchronization error $\boldsymbol{\delta}(k)$ is defined as:

$$\begin{aligned} \boldsymbol{\delta}(k) &= \mathbf{x}(k) - x^*\mathbf{1}_n = \mathbf{x}(k) - \frac{\mathbf{p}_w{}^T\mathbf{x}_0}{\mathbf{p}_w{}^T\mathbf{1_n}}\mathbf{1}_n \\ &= (\mathbf{I} - \frac{\mathbf{1}_n\mathbf{p}_w^T}{\mathbf{p}_w{}^T\mathbf{1_n}})\mathbf{x}(k), \end{aligned} \quad (7)$$

where $\mathbf{p}_w^T\mathbf{x}_0 = \mathbf{p}_w^T\mathbf{x}(k) = const.$

III. Gain Changing Consensus

If the system changes in time and the relationships between agents are changing, this can be described with weights varying over time $\omega_{ij}(k)$. We deal with a special case of general weight changing consensus, one where the topology is not directly considered switching (a_{ij} do not change over time), but only changes of elements of the gain matrix assumed; *gain-changing consensus*. Hence, adjacency, degree, gain and Laplacian matrices of the graph are all time dependant and can be written respectively: $\mathbf{A}_w(k)$, $\mathbf{D}_w(k)$, $\mathbf{\Gamma}(k)$, $\mathbf{L}_w(k)$.

The gain-changing consensus thus can be written as:

$$x_i(k+1) = x_i(k) + t_s \sum_{j\in\mathcal{N}_i} a_{ij}\gamma_{ij}(k)(x_j(k) - x_i(k)), \quad (8)$$

or in matrix form:

$$\mathbf{x}(k+1) = (\mathbf{I} - t_s\mathbf{L}_w(k))\mathbf{x}(k), \quad (9)$$

where $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity matrix and $\mathbf{x}(k) \in \mathbb{R}^n$. General convergence conditions of (9) are known [3] and are here fulfilled. However, we want to obtain conditions on gain matrix ($\mathbf{\Gamma}(k)$), graph topology ($\mathbf{A}$) and time step

(t_s) to ensure the convergence of the proposed method, which can be achieved by analyzing the synchronization error dynamics $\boldsymbol{\delta}(k)$.

Definition III.1. *[12] Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w(k)$ (4). Synchronization error $\boldsymbol{\delta}(k)$ is defined as :*

$$\boldsymbol{\delta}(k) = (\mathbf{I} - \frac{\mathbf{1_n}\boldsymbol{\pi}_w^T(k)}{\boldsymbol{\pi}_w^T(k)\mathbf{1_n}})\mathbf{x}(k), \tag{10}$$

where $\boldsymbol{\pi}_w(k)$ is a projection vector such that $lim_{k\to\infty}\boldsymbol{\pi}_w(k)$ is the left-zero eigenvector of $lim_{k\to\infty}\mathbf{L}_w(k)$ and $\boldsymbol{\pi}_w(k)^T\mathbf{x}(k) = const.$

The stability follows using Lyapunov arguments summarized here for convenience.

Lemma III.1. *[13] Consider a system (9) and assume that the graph associated with $\mathbf{L}_w$ is connected and bidirectional. Define a Lyapunov function $V(k, \boldsymbol{\delta})$ for the system as:*

$$V(k, \boldsymbol{\delta}) = \boldsymbol{\delta}(k)^T\mathbf{P}(k)\boldsymbol{\delta}(k), \tag{11}$$

where $\mathbf{P}(k)$ is a positive definite matrix. The following conditions must hold for the delta-origin to be stable:

$$\begin{aligned} &V(k, \boldsymbol{\delta}) = 0, \boldsymbol{\delta} = 0 \\ &V(k, \boldsymbol{\delta}) > 0, \forall \quad \boldsymbol{\delta} \neq 0 \\ &\Delta V(k, \boldsymbol{\delta}) = V(k+1, \boldsymbol{\delta}(k+1)) - V(k, \boldsymbol{\delta}) < 0, \forall k \end{aligned} \tag{12}$$

IV. Main Results

In this section we state the conditions guaranteeing the convergence of the proposed novelty-biased protocol to consensus using Lyapunov methods.

Theorem IV.1. *Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w(k)$ defined as in (4), its left-zero eigenvector $\mathbf{p_w}(k)$ and consensus protocol (9). If the elements of the left-zero eigenvector $\mathbf{p_w}(k)$ are monotonically decreasing (i.e. $\mathbf{p_w}(k+1) - \mathbf{p_w}(k) \preceq 0$) and the upper bound on the t_s is defined by:*

$$\frac{\sum_u a_{iu}\gamma_{iu}(k)p_{w_i}(k)}{\sum\limits_u a_{iu}\gamma_{iu}^2(k)p_{w_i}(k) + \sum\limits_u p_{w_u}(k)a_{iu}\gamma_{ui}^2(k)} \geq t_s \tag{13}$$

$$\begin{aligned} &\frac{f_1(k)}{f_2(k)} \geq t_s, \\ f_1(k) =& a_{ij}(\gamma_{ij}(k)p_{w_i}(k) + \gamma_{ji}(k)p_{w_j}(k)) \\ f_2(k) =& a_{ij}(\gamma_{ij}(k)p_{w_i}(k)\sum_u a_{iu}\gamma_{iu}(k) \\ &+ \gamma_{ji}(k)p_{w_j}(k)\sum_u a_{ju}\gamma_{ju}(k)) \\ &- \sum_u p_{w_u}(k)a_{ui}\gamma_{ui}(k)a_{uj}\gamma_{uj}(k), \end{aligned} \tag{14}$$

$\forall i, j \in \mathcal{V}$, then the system reaches consensus asymptotically.

Proof. Using the Lyapunov function (11) from Lemma III.1, with $\mathbf{P}(k)$ a positive diagonal matrix whose elements are elements of the left-zero eigenvector of $\mathbf{L}_w(k-1)$

$$\mathbf{P}(k) = \text{diag}(\mathbf{p_w}(k-1)) > 0. \tag{15}$$

The first two conditions in Lemma III.1 hold. To evaluate the third condition we can rewrite ΔV using (9):

$$\begin{aligned} \Delta V =& \boldsymbol{\delta}(k+1)^T\boldsymbol{P}(k+1)\boldsymbol{\delta}(k+1) - \boldsymbol{\delta}(k)^T\mathbf{P}(k)\boldsymbol{\delta}(k) \\ =& -\boldsymbol{\delta}(k)^T\mathbf{Q}(k)\boldsymbol{\delta}(k), \end{aligned} \tag{16}$$

where $\mathbf{Q}(k) = \mathbf{P}(k) - \mathbf{P}(k+1) + t_s(\mathbf{L}_w(k)^T\mathbf{P}(k+1) + \mathbf{P}(k+1)\mathbf{L}_w(k) - t_s\mathbf{L}_w(k)^T\mathbf{P}(k+1)\mathbf{L}_w(k)) = \mathbf{P}(k) - \mathbf{P}(k+1) + t_s\mathbf{N}(k)$. For system (9) to converge to a consensus, $\mathbf{Q}(k)$ needs to be positive definite, however it can also be positive semi-definite, with consensus subspace as its null space, because the definition of $\boldsymbol{\delta}(k)$ effectively makes it positive-definite [14]. If the elements of the $\mathbf{p_w}(k)$ are monotonically decreasing, then $\mathbf{P}(k) - \mathbf{P}(k+1) \geq 0$. Matrix $\mathbf{N}(k)$ is symmetric, because it consists of symmetric positive semi-definite part $(\mathbf{L}_w(k)^T\mathbf{P}(k+1) + \mathbf{P}(k+1)\mathbf{L}_w(k))$ [15] and symmetric negative semi-definite term $(-t_s\mathbf{L}_w(k)^T\mathbf{P}(k+1)\mathbf{L}_w(k))$. With the choice of $\mathbf{P}(k+1)$ to have elements of the left-zero eigenvector of $\mathbf{L}_w(k)$ on the diagonal, matrix $\mathbf{N}(k)$ has all row sums equal to zero, as well as its column sums. The elements of the matrix are then:

$$\begin{aligned} n_{ii} =& \sum_u a_{iu}\gamma_{iu}(k)p_{w_i}(k) - t_s(\sum_u a_{iu}\gamma_{iu}^2(k)p_{w_i}(k) \\ &+ \sum_u p_{w_u}(k)a_{iu}\gamma_{ui}^2(k)) \end{aligned} \tag{17}$$

$$\begin{aligned} n_{ij} =& t_s(a_{ij}\gamma_{ij}(k)p_{w_i}(k)\sum_u a_{iu}\gamma_{iu}(k) \\ &+ a_{ji}\gamma_{ji}(k)p_{w_j}(k)\sum_u a_{ju}\gamma_{ju}(k) \\ &- \sum_u p_{w_u}(k)a_{ui}\gamma_{ui}(k)a_{uj}\gamma_{uj}(k)) \\ &- (a_{ij}\gamma_{ij}(k)p_{w_i}(k) + a_{ij}\gamma_{ji}(k)p_{w_j}(k)). \end{aligned} \tag{18}$$

Under the conditions stated in Theorem IV.1, matrix $\mathbf{N}(k)$ is a Laplacian of a symmetric connected graph and by that it is positive semidefinite, which makes the matrix $\mathbf{Q}(k)$ positive semidefinite. The stability of the δ-origin is thereby guaranteed. □

V. Gain Matrix Adjustment

Consider a multi-agent system that has reached consensus on some value. After the agent i obtains a new information at the time step k_{o_i}, it changes state $x_i(k)$ to new value. This new value can be a result of an underlying algorithm (e.g. POMDP) based on this measurement or it may directly become this new measurement. It is defined by the use-case. To deviate from average consensus in a multi-agent system, at the time of the new measurement by an agent, one or more agents need to change their gain matrix. Using Theorem IV.1, we introduce bias towards the most recent information in two ways:

1) Node-based weight-changes: Agent i with the most recent information at the time k steers the final consensus value towards its measurement without

any additional communication. This is achieved by changing the row of the gain matrix as: $\gamma_{ij}(k) = g_i(k), \forall j \in \mathcal{V}$, where $\mathbf{g}(k) = [g_1(k), g_2(k), \cdots, g_n(k)]$. The agent i treats all neighbours' information equally.

2) Edge-based weight-changes: Agent i communicates that it has the newest measurement, and its neighbours steer the final consensus value towards the value from agent i. This is achieved by changing the column of the gain matrix as: $\gamma_{ij}(k) = g_j(k), \forall i \in \mathcal{V}$, where $\mathbf{g}(k) = [g_1(k), g_2(k), \cdots, g_n(k)]$. All neighbouring agents treat the information from agent i equally.

A. Node-based weight-changes

For $\gamma_{ij}(k) = g_i(k)$, the system (8) can be written as:

$$x_i(k+1) = x_i(k) + t_s g_i(k) \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) \quad (19)$$

Matrix $\mathbf{L}_w$ of this system is:

$$\mathbf{L}_w(k) = \begin{bmatrix} |\mathcal{N}_1| g_1(k) & -a_{12} g_1(k) & \cdots & -a_{1n} g_1(k) \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} g_n(k) & -a_{n2} g_n(k) & \cdots & |\mathcal{N}_n| g_n(k) \end{bmatrix} \quad (20)$$

The $\mathbf{L}_w$ matrix is not symmetric, i.e. $\omega_{ij} \neq \omega_{ji}$, but the property of zero row sum remains. However the property of zero column sum is not preserved, which makes the left-zero eigenvector: $\mathbf{L}_w(k)$: $\mathbf{p}_w^T(k)\mathbf{L}_w(k) = \mathbf{0}_n \longrightarrow \mathbf{p}_w(k) = [1/g_1(k), 1/g_2(k), 1/g_3(k), \cdots, 1/g_n(k)]^T$.

Theorem V.1. *Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w(k)$ defined as in (20) and the consensus protocol (9). If the gains $g_i(k)$ are monotonically increasing functions:*

$$g_i(k+1) - g_i(k) \geq 0, \quad (21)$$

and the time step t_s satisfies the bounds:

$$\frac{2|\mathcal{N}_i|}{g_i(k)|\mathcal{N}_i|^2 + \sum_{u \in \mathcal{N}_i} g_u(k)} \geq t_s, \quad (22)$$

$$\frac{2a_{ij}}{a_{ij}|\mathcal{N}_i| g_i(k) + a_{ij} g_j(k) |\mathcal{N}_j| - \sum\limits_{u \in \mathcal{N}_i \cap \mathcal{N}_j} g_u(k)} \geq t_s, \quad (23)$$

$\forall i, j \in \mathcal{V}$, then consensus is reached asymptotically.

Proof. It directly follows from the fact that we choose $\mathbf{P}(k)$ to have the elements of the left-zero eigenvector of Laplacian (20),$\mathbf{L}_w(k-1)$. The matrix $\mathbf{P}(k)$ is:

$$\mathbf{P}(k) = \text{diag}([1/g_1(k-1), 1/g_2(k-1), \cdots, 1/g_n(k-1)]), \quad (24)$$

where $g_i(k)$ is node (locally) changed gain parameter from (19). From Theorem IV.1, left-zero eigenvector needs to be monotonically decreasing, i.e. $1/g_i(k-1) - 1/g_i(k) \leq 0$, implying that gains need to be monotonically increasing: $g_i(k+1) - g_i(k) \geq 0$. If $\gamma_{ij}(k) = g_i(k)$ and $p_{w_i}(k) = 1/g_i(k)$ are substituted in (13) and (14), we obtain (22) and (23). □

B. Edge-based weight-changes

For $\gamma_{ij}(k) = g_j(k)$, the system (8) can be written as:

$$x_i(k+1) = x_i(k) + t_s \sum_{j \in \mathcal{N}_i} a_{ij} g_j(k)(x_j(k) - x_i(k)), \quad (25)$$

Matrix $\mathbf{L}_w$ of this system is:

$$\mathbf{L}_w(k) = \begin{bmatrix} \sum_{u \in \mathcal{N}_1} g_u(k) & -a_{12} g_2(k) & \cdots & -a_{1n} g_n(k) \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} g_1(k) & -a_{n2} g_2(k) & \cdots & \sum_{u \in \mathcal{N}_n} g_u(k) \end{bmatrix} \quad (26)$$

Similar to (20), this $\mathbf{L}_w(k)$ matrix is not symmetric, i.e. $\omega_{ij} \neq \omega_{ji}$, but the property of zero row sum remains. Also the property of zero column sum is not preserved, which makes the left-zero eigenvector depend on elements of the $\mathbf{L}_w(k)$: $\mathbf{p}_w^T(k)\mathbf{L}_w(k) = \mathbf{0}_n \longrightarrow \mathbf{p}_w = [g_1(k), g_2(k), \cdots, g_n(k)]^T$.

Theorem V.2. *Consider a connected, bidirectional graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with Laplacian matrix $\mathbf{L}_w(k)$ defined as in (26) and the consensus protocol (9). If the gains $g_i(k)$ are monotonically decreasing functions:*

$$g_i(k+1) - g_i(k) \leq 0, \quad (27)$$

and time step t_s satisfies the bounds:

$$\frac{2}{g_i(k) + \sum_{u \in \mathcal{N}_i} g_u(k)} \geq t_s \quad (28)$$

$$\frac{2a_{ij}}{a_{ij} \sum\limits_{u \in \mathcal{N}_i} g_u(k) + a_{ij} \sum\limits_{u \in \mathcal{N}_j} g_u(k) - \sum\limits_{u \in \mathcal{N}_i \cap \mathcal{N}_j} g_u(k)} \geq t_s, \quad (29)$$

$\forall i, j \in \mathcal{V}$, then consensus is reached asymptotically.

Proof. The steps of the proof are the similar to those in Section V-A. We define the matrix $\mathbf{P}(k)$ to have left-zero eigenvector of the (26) on the diagonal :

$$\mathbf{P}(k) = \text{diag}([g_1(k-1), g_2(k-1), \cdots, g_n(k-1)]), \quad (30)$$

where $g_i(k)$ is gain parameter from (25). From Theorem IV.1, left-zero eigenvector needs to be monotonically decreasing, i.e. $g_i(k-1) - g_i(k) \leq 0$, implying that the gains need to also be monotonically decreasing. Substituting $\gamma_{ij}(k) = g_j(k)$ and $p_{w_i}(k) = g_i(k)$ in (13) and (14), we obtain the expressions in (28) and (29). □

Note that by assigning gains we impact the left-zero eigenvector which defines final consensus value.

VI. Bias Introduction

In this paper, two specific classes of functions are considered to introduce the bias towards novel information, with respect to the convergence conditions on both types of weight-changes considered in the paper, (21) for node-based and (27) for edge-based.

A. Piecewise constant gains

Let $g_i(k)$ be defined as follows:

$$g_i^c(k) = \begin{cases} c_1 & , \ k_{o_i} < k < k_c \\ c_2 & , \ \text{otherwise}, \end{cases}, \tag{31}$$

where k_{o_i} is the time step at which measurement o_i is acquired by the agent i, and k_c is the time-step at which consensus is considered to be reached.

In this case, the value of $g_i(k)$ is switching between two constant positive values. Depending on choice of c_1 and c_2, with $c_1 \neq c_2$, one can observe that either at gain-switching time k_{o_i} or k_c the function $g_i^c(k)$ will not satisfy the monotonically increasing or decreasing condition. However, at the very next step $g_i^c(k) = g_i^c(k-1)$, which satisfies both (21) or (27), meaning that the system is switching between two stable states, and according to [16], the overall switching system is also stable.

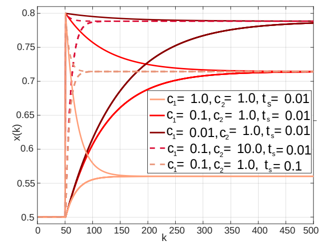
To introduce bias in node-based weight-changes, after the agent i obtains new information, it decreases the influence of the other agents' values. For $g_i^c(k)$, this is achieved by $c_1 < c_2$, and the responses for various permissible combinations of c_2 and c_1 are shown in Figure 1(a). In the edge-based case, after one of the agents obtains new information, the other agents value the new information the most by decreasing the influence of the values from others. In case of $g_i^c(k)$, this is achieved with $c_1 > c_2$. The response of such system is shown in Figure 2(a) for different values of c_1 and c_2.

B. Piecewise exponential gains

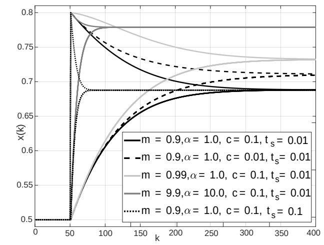
Let $g_i^e(k)$ be defined as:

$$g_i^e(k) = \alpha + m e^{-c(k-k_{o_i})t_s}, \tag{32}$$

where α, m and $c > 0$ are parameters of the function, and k_{o_i} is the time step at which the agent i received its latest measurement. Exponential function also satisfies the monotonically increasing/decreasing condition. The benefit of $g_i^e(k)$ over $g_i^c(k)$ is that it does not require checking if the consensus is reached. Similar to the $g_i^c(k)$, at the time of new measurement k_{o_i}, the condition of monotonic function may not be satisfied. At the very next time step, the monotonic properties of $g_i^e(k)$ will be reestablished and the overall system will be stable since we are again switching between two stable systems. In the node-based case, $g_i^e(k)$ needs to be monotonically increasing which implies $m < 0$, and since gains are defined to be non-negative, we need to ensure $\alpha > m$. The convergence of such consensus system for various combinations of α and m are shown in Figure 1(b). Function $g_i^e(k)$ in the case of edge-based weight-changes needs to be monotonically decreasing, which can be ensured with $m > 0$. The convergence of such consensus system for various combinations of α and m are shown in Figure 2(b). Comparing Figures 1 and 2, we can observe that changing the parameters of $g_i^c(k)$ and $g_i^e(k)$ we can affect both the final value of the biased consensus and the

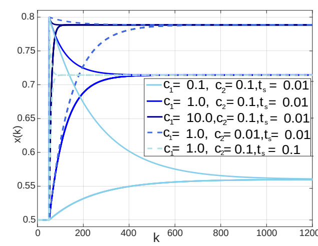


(a) Gain changes according to piecewise constant gains $g_i^c(k)$

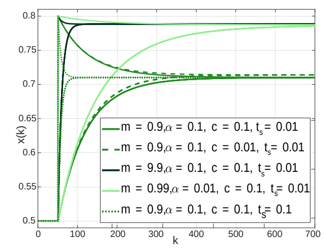


(b) Gain changes according to piecewise exponential gain $g_i^e(k)$

Fig. 1: Node-based consensus: 5 fully connected agents.



(a) Gain changes according to piecewise constant gains $g_i^c(k)$



(b) Gain changes according to piecewise exponential gain $g_i^e(k)$

Fig. 2: Edge-based consensus: 5 fully connected agents.

speed of convergence towards the final value, while keeping the system stable. Our initial analysis of final values and rate of convergence show significant differences between both gain-changing approaches (node vs. edge change) and gain-changing functions which shows the versatility of the proposed novelty-biased consensus framework.

VII. Use Case Example

To showcase the deployment of the proposed method, we use a simple problem called *Fire on a grid* [17].

The problem is defined on 2×2 grid, where two agents must locate the fire. Fire states represent the location of the fire on the grid and they form a vector that contains beliefs of the location of fire. The main goal here is to end the mission when the fire is found and agents are sure they have sufficient information about its location. Fire states are extended by an additional state which is used to track the progress of information gathering, the state *unknown*. Therefore the total belief vector of one agent consists of 5 elements. Agents start the search with a large belief that the location of the fire is unknown. During the search the belief over this state should decrease. They take actions of moving on a grid and action that terminates the search for the fire, when they are sure of the location of the fire. The search is successful if the maximum value in belief states of fire coincides with the correct location of the fire on the grid. In the Figure 3 we can see the belief evolution, using novelty-biased consensus and underlying decision-making framework (POMDP), where the states $\mathbf{x}(k) \in [0,1]^{(2\times5)}$. The node-based weight-changes in the consensus of the form (9) with constant gains ($c_1 = 0.01, c_2 = 1.0$) and $t_s = 0.1s$ is employed. The state is described as a vector, but the consensus is reached component-wise on state vector of agents. Each time an agent visits the location, it

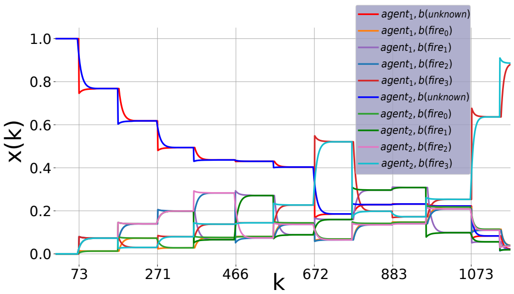


Fig. 3: The evolution of the consensus value in the Fire on a grid problem [17]. Agents make decisions using a POMDP and node-based consensus. Abrupt changes of $x(k)$ indicate new measurements followed by belief sharing.

obtains reading from sensors, which can be faulty. After the sensor reading, it reforms its belief vector based on the underlying POMDP model, and creates the new belief distributions of possible locations of the fire. The peaks represent this new belief, after the measurement from the sensor. This new belief is then shared with the other agent throughout the novelty-biased consensus and the final value is biased towards the more recent belief. We can see that as they are exploring the environment, the belief of the unknown location is dropping, and others are increasing. At $k = 770$, the agent 2 obtained wrong measurement, as at $k = 883$ also did the agent 1. If the consensus on belief were not included, both of the agents could have concluded wrong location of the fire. Including the bias, accelerates final conclusion, without too much degradation of the new information.

VIII. Conclusion

In this paper we presented distributed protocol based on a consensus algorithm, where agents reach the common value biased towards the most recent information in the system. Two main approaches are introduced and their stability ranges are provided from sufficient conditions. For each approach, piecewise constant and piecewise exponential gain-changing functions are proposed and their parameters are selected to achieve the bias towards novel information while ensuring stability and convergence of the consensus protocol. Simulations indicate that the parameters of these gain-changing functions can be used to control the final value and the speed of convergence towards that value. The future work involves deriving the conditions without the constraint of bidirectional graphs and exploring other time-varying monotonically decreasing/increasing functions as well as convergence rates for proposed gains, where the work in [12] may provide valuable insights. Additionally the agents with different sensor qualities will be included, with the gains adjusted accordingly. Sensor quality not only impacts the consensus, but the whole model of POMDP. Therefore, in our ongoing POMDP research we aim to find the minimum quality of sensors threshold (percentage of correct measurement) at which the combination of novelty-biased consensus and POMDP in multi-agent systems ceases to maintain cooperative behaviour.

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