A Note on Treatment of Convexly Non-Liftable Polyhedral Partitions

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Abstract— **Piecewise affine convex liftings prove to be versatile tools for the analysis and implementation of piecewise affine control systems. This paper contributes to the theory of convex lifting of polyhedral partitions, by introducing a more efficient approach to the treatment of convexly non-liftable partitions or subpartitions. The main contribution is a new algorithmic procedure to refine polyhedral partitions that do not admit a convex lifting, into convexly liftable ones. This proves the potential computational advantages of a precise, local focus on the problematic regions while preserving the internal boundaries of the original partition to be lifted.Additionally, an algorithm to identify those problematic regions is also introduced and some operations with convexly liftable partitions are discussed.**

I. Introduction

Convex lifting is one of the many transformative mathematical approaches that can be used to effectively address complex analysis and design problems, spanning from computational geometry, computer graphics, where convex liftings can support advanced rendering techniques by enabling efficient mesh manipulation, to mechanics and control [1]. In particular for control related topics, convex liftings can play the role of candidate Lyapunov functions [2], can be used as indicator function for active piecewise control implementation [3], control design for hybrid systems [4] or support the navigation strategies in cluttered environments [5]. In this paper we focus specifically on the polyhedral convex lifting, equivalently understood as a convex, continuous, piecewise affine (PWA) function defined over a polyhedral partition such that any pair of different regions is lifted onto two distinct hyperplanes. It can enhance the precision and manageability of PWA functions, making it a critical tool for optimization [6], infeasibility detection [7], control laws complexity reduction [8], and computational geometry [9].

The authors in [10] provide a comprehensive overview of the concept of polyhedral convex lifting and present algorithms for constructing convex liftings of polyhedral partitions. These algorithms solve a convex optimization problem and their feasibility also serves as a necessary and sufficient condition for the convex liftability of the polyhedral partition. The certification of convex liftability is thus essential for constructing a convex lifting, which can in turn be exploited in various control design and implementation techniques. Furthermore, the use of polyhedral convex lifting can be extended to partitions that are convexly non-liftable in the original configuration after a (finite) sequence of geometric transformations. We focus in the present paper on the specific case of polyhedral partitions that can not be convexly lifted in their original configuration. These partitions will be referred to as *convexly non-liftable*, or simply *non-liftable partitions*.

It is commonly accepted that the study of a polyhedral convex lifting is well-possed within the class of cell complexes. A cell complex is a polyhedral partition that satisfies the face-to-face property, where the intersection of any two regions is either empty or a shared lower-dimensional face [11]. Rybnikov [12] proposed a detailed discussion of equivalent conditions for convex liftability. For concision, two structurally distinctive conditions can be recalled: *A given cell complex in $\mathbb{R}^d$ possesses a convex lifting if and only if one of the following holds:*

- *it admits a strictly positive d-stress,*
- *it is an additively weighted Dirichlet–Voronoi diagram or Delaunay decomposition.*

While stresses in planar structures describe the distribution of forces in equilibrium, d-stresses extend this notion to higher-dimensional polyhedral frameworks. This allows for the study of equilibrium conditions and liftability in more complex, multi-dimensional geometric structures, building on the isomorphism between liftings and stresses, known as the Maxwell correspondence.

The work [10] also addresses polyhedral partitions whose convex liftability is not fulfilled. To achieve convex liftability of such partitions, rearranging by means of subcutting the polyhedral regions is possible. However, in many applications the initial boundaries have to be maintained too, in order to preserve the original structure of the problem. In [10] it is proved that there exists at least one subdivision (preserving the initial boundaries of this partition), such that the new cell complex is convexly liftable.

This paper will elaborate on this topic, exploring solutions for this subdivision, particularly focusing on those techniques that could potentially reduce the number of regions within the partitions after the refinement process.

Notions

Before proceeding further, let us briefly recall some definitions and notions used henceforth. A polyhedron is defined as the intersection of finitely many closed halfspaces. A bounded polyhedron is called a polytope. In this paper, we will be dealing only with bounded polyhedra. For a given

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polyhedron $\mathcal{S} \subseteq \mathbb{R}^d$, we use $\mathcal{V}(\mathcal{S})$ to denote the set of its vertices. If $\mathcal{S} \subseteq \mathbb{R}^d$ constitutes a polyhedron of full dimension in the space, its faces are the intersections formed by $\mathcal{S}$ and its supporting hyperplanes. More generally, a k-face denotes a face of $\mathcal{S}$ with a dimensionality of $k \leq d$. For instance, a 0-face is identified as a vertex, a 1-face as an edge, and a $(d-1)$-face as a facet. Additionally, $\mathcal{F}(\mathcal{S})$ denotes the collection of all facets belonging to the polyhedron $\mathcal{S}$.

II. Preliminaries and problem statement

The central notion of the current work is represented by the positive definite functions with domain of definition represented by unions of polyhedral sets. In the following, we use the notation $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ to represent such collection of polyhedra $\mathcal{X}_i \subset \mathbb{R}^d$ with the index i belonging to a finite ordered set $\mathcal{I}_N = \{1, 2, \dots, N\} \subset \mathbf{N}$.

Definiton 1: A collection of $N \in \mathbb{N}_{>0}$ full-dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^d$, denoted by $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^d$ if

1) $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$;
2) $\text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_i) = \emptyset$ where $i \neq j, (i,j) \in \mathcal{I}_N^2$, $i \neq j$.

Also, if $\mathcal{X}$ is a polytope, then $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ is called a *polytopic partition*.

Two polyhedra $\mathcal{X}_i$ and $\mathcal{X}_j$ within a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ can exhibit a non-empty intersection along their boundaries. These closed sets are *neighbors* if

$$\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d - 1.$$

The term *cell complex*, refers to a specific class of polyhedra, similar to the notation in [13]:

Definiton 2: A polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is a *cell complex* in $\mathbb{R}^d$ if for any pair of neighboring $\mathcal{X}_i, \mathcal{X}_j$ where $i, j \in \mathcal{I}_N$ is the $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$ a face of both $\mathcal{X}_i$ and $\mathcal{X}_j$.

In other words, a *cell complex* is a polyhedral partition whose face-to-face property is fulfilled.

We refer to a subset of polyhedra of a given partition as *subpartition*. Specifically, for a partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ if $\mathcal{J}$ is a subset of $\mathcal{I}_N$ then $\{\mathcal{X}_i\}_{\mathcal{J}}$ is a partition of $\bigcup_{i \in \mathcal{J}} \mathcal{X}_i$ and a subpartition of $\mathcal{X}$.

Definiton 3: A *hyperplane arrangement* $\mathcal{A} \in \mathbb{R}^n$ is a finite collection of affine hyperplanes, which are $(n-1)$-dimensional subspaces of $\mathbb{R}^n$.

In the context of this work, hyperplane arrangement refers to the collection of all hyperplanes defining the polyhedra of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. By extension [14], a refined partition of $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ can be generated by all possible intersections of half-space related to the half-space arrangements. With an abuse of notation but without ambiguity in the context, such a partition will be also denoted as hyperplane arrangement.

Definiton 4: Given the partition $\mathcal{X} = \bigcup\limits_{i \in \mathcal{I}_N} \mathcal{X}_i$, the function

$$z(x) = a_i^\top x + b_i \text{ for any } x \in \mathcal{X}_i$$

is called a *convex* (piecewise affine) *lifting* if the following conditions hold true:

1) $z(x)$ is continuous over $\mathcal{X}$;
2) for each $i \in \mathcal{I}_N, z(x) > a_j^\top x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $i \neq j, j \in \mathcal{I}_N$.

We now revisit the definition of equilibrium stresses, as described in [15], [12] and also recalled in [16]. For clarity, let $\overrightarrow{n}(F, \mathcal{S})$ represent the inward unit normal vector to the polyhedron $\mathcal{S}$ at its facet F, meaning the unit vector $\overrightarrow{n}(F, \mathcal{S})$ is perpendicular to F and points inward toward $\mathcal{S}$. Additionally, the star of a face in a cell complex $\mathcal{S}$ refers to the smallest subcomplex that contains all the faces of $\mathcal{S}$ which include this face.

A real-valued function $s(.)$ defined on the $(d-1)$-faces of a cell complex $K \subset \mathbb{R}^d$ is called a d-stress if at each internal $(d-2)$-face F of K:

$$\sum_{\mathcal{S} | F \subset \mathcal{S}} s(S) \overrightarrow{n}(F, \mathcal{S}) = 0, \tag{1}$$

where this sum ranges over all $(d-1)$-faces in the star of F (the $(d-1)$-faces such that F is their common facet). The quantities $s(\mathcal{S})$ are the coefficients of this d-stress, are called a tension if the sign is strictly positive, and a compression if the sign is strictly negative.

If a convex lifting fails to exist over the partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$, then for any candidate piecewise affine function, the second condition in Definition 4 is violated for a subset of polyhedra. We will refer to them as *non-liftable regions* or in a less formal manner as *problematic polyhedra*.

Recalling the necessary and sufficient conditions for the existence of piecewise affine convex liftings [12], it is clear that positive d-stresses cannot be guaranteed on the facets of the problematic polyhedra.

The subsequent developments will employ the notion of *partial lifting* defined as piecewise affine lifting over the partition excluding the problematic polyhedra. Formally, this is done by restricting the domain of the lifting function.

The next sections aim to address the following objectives:

- revisit the hyperplane arrangements as a tool to mitigate the non-liftable regions in a polytopic partition with a focus on the complexity;
- construct an algorithmic procedure to identify the non-liftable regions
- analyse the strategies for local treatment of non-liftability, with the aim of certifying the global liftability.

III. Global refinement strategies exploiting hyperplane arrangement

The necessary and sufficient conditions discussed in Sec. I intuitively suggest geometrical approaches to modify existing convexly non-liftable partitions, and enable liftability, while preserving the original structure of the partition. Typical modifications involve sub-cutting, or more precisely refining, the existing partition to attain additively weighted Delaunay triangulations, Dirichlet–Voronoi diagrams [17], or reconstructing the structure with equilibrium d-stresses all by maintaining the initial boundaries.

While Delaunay triangulations and Dirichlet–Voronoi diagrams are difficult to extend in higher dimensions, the

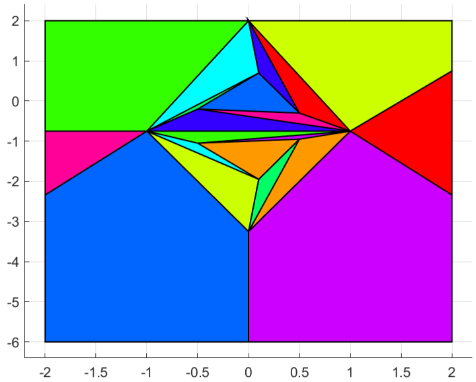
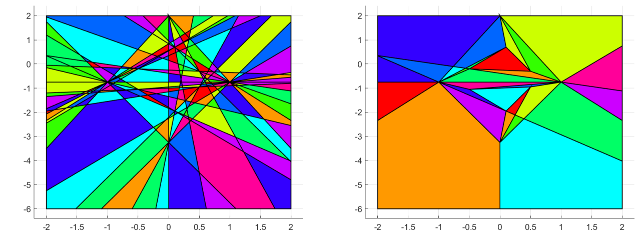


Fig. 1: An illustrative example of a convexly non-liftable cell complex in $\mathbb{R}^2$ consisting of 20 polyhedra.



(a) (b)

Fig. 2: Illustrative examples of refined partitions. (a) Cell complex resulted from Alg. 4 in [10] – 250 polyhedra. (b) Cell complex resulted from refinement using a random hyperplane selection from problematic polyhedra – 28 regions.

approach involving equilibrium d-stresses is straightforward to generalize. Therefore, we focus our further discussion in this realm.

The refinement methods that address the entire partition can be labelled as global approaches. For instance, the technique discussed in [17] refines the partition to create a Voronoi diagram. Another example is the use of hyperplane arrangement in [10], where the hyperplanes defining the partitions polyhedra are used to decompose the domain. This technique subdivides the non-liftable partition to achieve liftability by restoring equilibrium d-stresses on the vertices.

This section explores potential global solutions for reducing the number of polyhedra in a convexly liftable partition, particularly using the hyperplane arrangement methods [18].

Although the hyperplane arrangement has the advantage to be essentially constructive independent on the dimension it necessitates substantial computational resources due to its combinatorial principles. The high demand for computational resources is generally manageable, taking into account that the geometrical refinement and the lifting computations can be performed offline. However, it is important to note that hyperplane arrangement provides an upper bound in terms of regions within the polyhedral partition. Consequently, the potential number of unnecessary polyhedra subcutting within the resulting partition impacts the real-time computational load when employing the resulting convex lifting tools in control [3]. Fig. 1 illustrates a cell complex, where the convex non-liftability is a result of the presence of simplex (triangle) regions which doesn't satisfy d-stress property. Fig. 2 shows its liftable refinement achieved on one side through a complete (exhaustive) hyperplane arrangement and, as a term of comparison, a partial sub-cutting using supporting hyperplanes of the non-liftable (simplex) regions. This simple comparison illustrates the notable increase in the number of polyhedra as resulting from a complete refinement process and the benefit of a *partial* or *local* refinement.

A natural question arises: *Is there a systematic procedure to generate refined partitions in* $\mathbb{R}^d$ *with guarantees of liftability but with a reduced number of polyhedra in comparison with the hyperplane arrangements ?*

Let us denote the set of hyperplanes used in the (non-redundant) definition of the regions $\{\mathcal{X}\}_{\mathcal{I}_N}$ as $\mathcal{H}(\{\mathcal{X}\}_{\mathcal{I}_N})$. This set can be rewritten as the disjoint union:

$$\mathcal{H}(\{\mathcal{X}\}_{\mathcal{I}_N}) = \mathcal{H}_0 \cup \mathcal{H}_a$$

with $\mathcal{H}_0$ collecting the hyperplanes $h \in \mathcal{H}_0$ for which

$$\begin{aligned}\operatorname{card}(\{\mathcal{X}\}_{\mathcal{I}_N} \cap \{x : h(x) \leq 0\}) \\ + \operatorname{card}(\{\mathcal{X}\}_{\mathcal{I}_N} \cap \{x : h(x) \geq 0\}) > N.\end{aligned}$$

Proposition 1: If $\mathcal{H}_a = \mathcal{H}$, then the partition $\{\mathcal{X}\}_{\mathcal{I}_N}$ is liftable.

Proof: This proposition holds true, as the use of all the defining hyperplanes for the decomposition corresponds to the hyperplane arrangement, which inherently results in a convexly liftable partition, as demonstrated in [10].

The increased number of polyhedra after the treatment by the hyperplane arrangement arises from its utilization of all hyperplanes $\mathcal{H}(\{\mathcal{X}\}_{\mathcal{I}_N})$, resulting in numerous unnecessary sub-cutting of the existing regions. Intuitively, reducing the number of hyperplanes used for refinement decreases the total number of regions. One straightforward strategy involves selecting only $\mathcal{H}_a \subset \mathcal{H}$, rather than using all hyperplanes. Probably the first idea coming to one's mind is to do so by randomly picking hyperplanes and iteratively test the convex liftability of the refined partition, utilizing the construction feasibility outlined in [10]. This approach is described in Alg. 1.

Utilizing Alg. 1 can significantly reduce the number of polyhedra in the refined partition, as demonstrated in the first row of Tab. I. The table displays the minimum, maximum, and mean numbers of polyhedra after 100 randomized addition orders. It also includes the original number of polyhedra in row 4 and the count after applying hyperplane arrangement in row 5. However, the number of resulting polyhedra for this approach is not solely determined by the partition's structure; it also depends on the sequence of adding hyperplanes. This can lead to a wider range in the final number of partitions. By introducing more information into the hyperplane selection process one may mitigate this uncertainty.

Let us consider a non-liftable partition $\{X\}_{I_N}$ containing a specific subpartition $\{X\}_{I_C} \subset \{X\}_{I_N}$, which prevents the

Algorithm 1 Algorithm for iterative refinement for a given polyhedral partition.

Input: Convexly non-liftable partition $\Omega = \{\mathcal{X}\}_{\mathcal{I}_N}$ in $\mathbb{R}^d$.
Output: Convexly liftable cell complex $\tilde{\Omega} = \{\mathcal{Y}\}_{\mathcal{I}_M}$.

```
G = [ ]
for i = 1 : N do
    X_i = {x : R_i x ≤ K_i}, G = [G; R_i K_i]
end for
Remove redundant rows of matrix G and randomize the order of the rows.
for i = 1 : size(G,1) do
    Ω̃ = ∅
    for j = 1 : |Ω| do
        X_j = {x : R_j x ≤ K_j}
        Y^(1) = {x : [R_j ; G(i,1:d)] x ≤ [K_j ; G(i,d+1)]}
        Y^(2) = {x : [R_j ; −G(i,1:d)] x ≤ [K_j ; −G(i,d+1)]}
        if dim(Y^(1)) = d & dim(Y^(2)) < d then
            Ω̃ ← Ω̃ ∪ {Y^(1)}
        else if dim(Y^(1)) < d & dim(Y^(2)) = d then
            Ω̃ ← Ω̃ ∪ {Y^(2)}
        else if dim(Y^(1)) = d & dim(Y^(2)) = d then
            Ω̃ ← Ω̃ ∪ {Y^(1), Y^(2)}
        end if
    end for
    Ω ← Ω̃
    Test the convex liftability of Ω̃ [10]
    if Ω̃ is convexly liftable then
        break
    end if
end for
```

construction of a convex lifting, e.g. the sliver triangles in the example presented in Fig. 1. This means that the regions from this problematic subpartition contain facets where positive d-stress cannot be sustained within the partition structure. Utilizing only the hyperplanes that define these problematic partitions would restore liftability of these subpartitions, by relaxing the d-stresses on the vertices of the involved regions. On the other hand, treating only those problematic subpartitions would lead to losing the cell complex property, therefore the refinement has to be extended to the remaining parts of the partition in order to certify global liftability.

Lemma 1: Refining all regions of a given liftable partition by any hyperplane maintains the liftability property for the resulting partition.

Proof: Let $\{X\}_{I_N}$ a partition in $\mathbb{R}^d$. We will show that the decomposition of $\{X\}_{I_N}$ by any hyperplane $\mathcal{H} \in \mathbb{R}^d$ leads to a new partition $\{X\}_{I_M}$, that is also liftable. The convex liftability of such a decomposition can be proven by employing the concept of stresses.

As $\{X\}_{I_N}$ is convexly liftable, for all internal $(d-2)$ –faces F_0 of $\{X\}_{I_N}$ it holds that

$$\sum s(F_i^{(d-1)}) n(F_0, F_i^{(d-1)}) = 0. \quad (2)$$

Then, by adding a new hyperplane $\mathcal{H}$ that has a not empty intersection with $\{X\}_{I_N}$, a refined partition $\{X\}_{I_M}$ is built. All internal $(d-2)$ –faces of $\{X\}_{I_M}$ intersecting with the hyperplane $\mathcal{H}$ are either faces that already existed in $\{X\}_{I_N}$, or new faces appearing on the intersection of the $\mathcal{H}$ with any $\mathcal{X}_i, i \in I_N$. (Internal $(d-2)$ –faces of $\{X\}_{I_M}$ not intersecting $\mathcal{H}$ fulfill (2) because of the convex liftability of $\{X\}_{I_N}$.)

For both cases, there exists a unique pair of $F_i^{(d-1)} \neq F_j^{(d-1)}$ such that $F_i^{(d-1)}$, $F_j^{(d-1)}$ lie in the common hyperplane $\mathcal{H}$ and have a common facet F_0. Accordingly, it can be seen that the inward unit normal vectors to the faces $F_i^{(d-1)}$, $F_j^{(d-1)}$ at their common facet F_0, denoted by $n(F_0, F_i^{(d-1)})$, $n(F_0, F_j^{(d-1)})$ respectively, satisfy

$$n(F_0, F_i^{(d-1)}) = -n(F_0, F_j^{(d-1)}).$$

Therefore, a pair of coefficients of strictly positive stresses $s(F_i^{(d-1)})$, $s(F_j^{(d-1)})$ exists (e.g. $s(F_i^{(d-1)}) = s(F_j^{(d-1)}) = 1$) such that

$$s(F_i^{(d-1)}) n(F_0, F_i^{(d-1)}) + s(F_j^{(d-1)}) n(F_0, F_j^{(d-1)}) = 0.$$

Now, applying the same argument for all the elements of F, one can obtain a strictly positive d-stress such that F_0 is in equilibrium. ■

Lemma 1 can also be easily extended to partial liftings, and then by adding all hyperplanes defining the problematic regions, the refined original structure can be restored.

TABLE I: Comparison of iterative algorithms for decomposing convexly non-liftable cell complexes.

	Algorithm	Number of polyhedra Min	Max	Mean	Ref.
1	Random hyperplane selection (Algorithm 1)	28	119	58	–
2	Iterative refinement with testing	46	73	59	–
2	Random hyperplane selection from problematic polyhedra	28	102	48	–
4	Original cell complex	–	–	–	20
5	Hyperplane arrangement	–	–	–	250

Addressing the non-liftability this way potentially reduces the overall number of regions in the refinement. While it affects the original partition globally, it moves toward resolving the issue of non-liftability on a local level. For this approach, the problematic regions have to be identified. However, locating these specific polyhedra in advance presents another challenge, although it can be avoided by addressing the issue iteratively. The algorithm's feasibility in constructing convex liftings itself is not confined to convex partitions; they need not even be connected or devoid of holes. This property does not impact the assessment of convex lifting feasibility based on the construction algorithm from [10]. Consequently, it becomes feasible to iteratively include regions and test their liftability. Whenever a subpartition becomes non-liftable, the hyperplanes of the last included polyhedron are employed to break down the original partition. The results of the refinements following these steps are included in the second row of Tab. I, similarly from 100 randomized tests.

Alternatively, if the set of regions causing the non-liftability was identified formerly, the hyperplanes refining the partition can be chosen only from the subset of hyperplanes defining the problematic regions, leading to different results, also reported in Tab. I, third row.

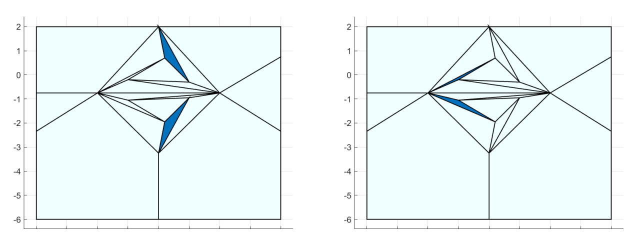
IV. Identifying non-liftability-inducing regions

Whenever a partition is non-liftable, a split can be performed resulting in two subpartitions: one liftable and the other nonliftable. This splitting is not unique, but the resulting subpartitions are informative in the process of retrieving the liftability.

Identifying the local causes of non-liftability by isolating a subset of regions, that prevent the construction of a convex lifting allows for a localized approach to the problem. Therefore, the avenues for identifying these problematic subsets are discussed in this section. These rely on the algorithm for constructing convex liftings from [10] as a tool for testing the liftability and exploiting the infeasiblity of the corresponding set of constraints.

The first approach involves iteratively adding regions and testing the liftability of the partition at each step without refining the partition, by only labeling problematic regions. Attempting this in one step would result in at most two subpartitions, a liftable one and a non-liftable one, since any region added after the first one that disables liftability will render the partition non-liftable. Therefore, if liftability is not achieved, the last added region is excluded and labelled as problematic before continuing. Similarly, removing regions and testing for liftability at each step works analogously.

It is worth emphasizing that the non-liftability occurs in internal stars, where equilibrium d-stresses can not be maintained. Consequently, this approach can identify different local sources of non-liftability for the same original partition, based on the order in which the regions are added, as it is illustrated in Fig. 3 for the non-liftable partition introduced in Fig. 1.



(a) (b)

Fig. 3: Examples of differently identified non-liftable subpartitions using the iterative construction method.

Another approach that we introduce here involves using a modified version of the convex lifting construction algorithm from [10]. The original algorithm efficiently determines the parameters of the hyperplanes that define the convex lifting by minimizing a suitable cost function subject to continuity and convexity conditions. Convex lifting obtained in this way ensures the absence of negative stresses. Adapting this optimization algorithm and introducing new parameters allowing for the inclusion of negative d-stresses, a tool for identifying problematic regions is obtained. This modification leads to a lifting, which is not strictly convex, exhibiting negative stresses of the structure. The presence of a negative d-stress between two regions indicates that the stress between them needs to be relaxed, meaning these regions may belong to the problematic partition, denoted $\{X\}_{I_C}$. For identifying all the problematic pairs of regions, the only task is to find the ones, connected by a negative stress. This algorithm is listed below as Alg. 2

Algorithm 2 Identification of problematic regions $\{\mathcal{Y}\}_{\mathcal{I}_M}$ for a given cell complex $\{\mathcal{X}\}_{\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^d$.

Input: Convexly non-liftable partition $\Omega = \{\mathcal{X}\}_{\mathcal{I}_N}$ in $\mathbb{R}^d$.
Output: Set of convexly non-liftable subpartitions $\{\mathcal{X}\}_{I_C}$

Register all pairs of neighboring regions in $\{\mathcal{X}\}_{\mathcal{I}_N}$.
for each pair of neighboring regions $(\mathcal{X}_i, \mathcal{X}_j)$ **do**
  add continuity conditions $\forall v \in \mathcal{V}(\mathcal{X}_i \cap \mathcal{X}_j)$

$$a_i^\top x + b_i = a_j^\top x + b_j \tag{3}$$

end for
for each pair of regions $(\mathcal{X}_i, \mathcal{X}_j)$ **do**
  add relaxation conditions $\forall v \in \mathcal{V}(\mathcal{X}_i) \setminus \mathcal{V}(\mathcal{X}_j)$

$$a_i^\top x + b_i \geq a_j^\top x + b_j + \epsilon_{i,j} \tag{4}$$

end for
Solve the following convex optimization problem by minimizing a chosen cost function, e.g.,

$$\min \Sigma(\epsilon_{i,j}^\top \epsilon_{i,j}) \text{ s.t. } (3), (4). \tag{5}$$

Find all regions connected by $\epsilon_{i,j} < -\varepsilon$.

Using this algorithm enables the decomposition of a given partition into convexly liftable and non-liftable subpartitions. The determination of the non-liftable subpartition relies on the evaluation of $\epsilon_{i,j}$ from the relaxation condition (4). In other words, if $\epsilon < 0$ between two regions, these regions are assigned to the non-liftable subset. A tolerance ε must be introduced for numerical robustness. However, the inclusion of this tolerance introduces variability and can affect the results of the decomposition. The specific values of ϵ and the choice of the cost function in the optimization process directly influence the decomposition as shown in Fig. 4, where different subpartitions are marked with distinct colors. These subpartitions represent neighboring regions connected by a $\epsilon_{i,j} < -\varepsilon$. The subsets identified as liftable consistently exhibit liftability across different setups.

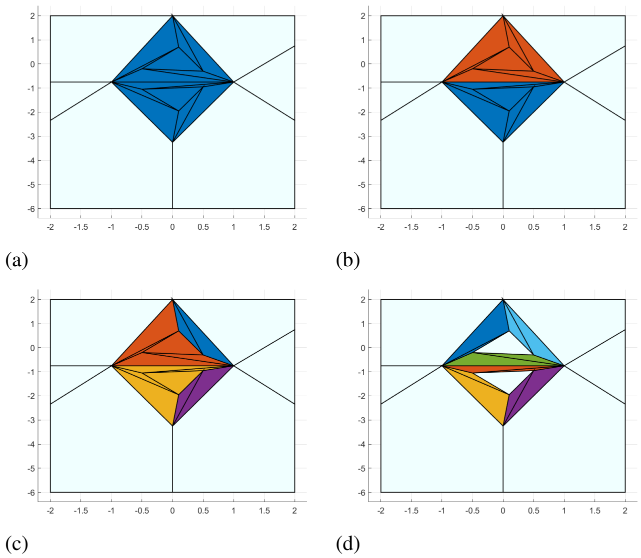


Fig. 4: Examples of differently identified non-liftable subpartitions based on the relaxation condition using different values of the numerical threshold ϵ.

Note that Alg. 2 relies on the vertex representation of the respective polyhedra, but can be modified to one based on their halfspace representation, analogously as in [10].

Whether all the hyperplanes defining the problematic regions $\mathcal{H}_a \subset \mathcal{H}$ are used for decomposition or only a subset of them, this approach does not guarantee achieving the minimal number of polyhedra. It however helps to reduce their average number, as demonstrated in Tab. I, third row.

Generally speaking, simplifying the refinement of convexly non-liftable partitions will be influenced by the focus of the approach:

- reduce the number of refinement hyperplanes,
- reduce the number of regions to be decomposed.

While these objectives are interconnected, the previously introduced approaches mainly focus on reducing the number of decomposing hyperplanes. Shifting the focus toward minimizing the number of regions represents a change of paradigm by addressing the convex non-liftability locally.

V. From Global to Local Treatment of Non-Liftability

This section discusses approaches that address the problem of non-liftability locally, specifically focusing on the previously identified problematic subpartitions while potentially introducing new hyperplanes. Treating problematic subpartitions locally involves treating each connected subpartition individually, without impacting those previously identified as liftable. While this approach does not necessarily reduce the number of resulting polyhedra—since it depends on the overall structure of the partition and the internal structure of each non-liftable subpartition—it offers the advantage of not requiring a complete precomputation. In certain applications, see e.g. [19], this allows to reuse some calculations.

Although methods based on the hyperplane arrangement (discussed in Sec. III) were shwown to lead to varying numbers of polyhedra in the final partition, leveraging existing hyperplanes consistently ensures that their number is never greater than what is achieved through standard hyperplane arrangements while obtaining convex liftability. However, introducing new hyperplanes does not automatically induce this property, even when these hyperplanes decompose only a subset of the regions. Therefore, it is essential to develop methods for combining the subpartitions labelled as liftable with the refined non-liftable subpartitions; thus ensuring the reconstruction of a convexly liftable partition that preserves the original structure.

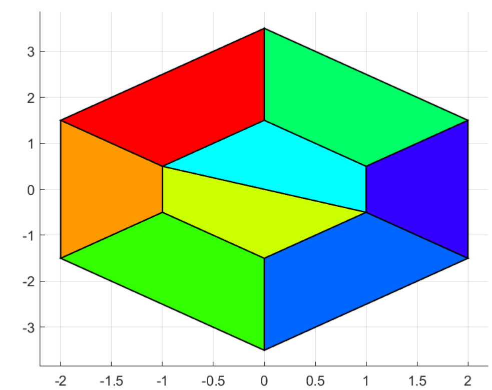


Fig. 5: A simple example of a convexly non-liftable cell complex in $\mathbb{R}^2$ consisting of 8 polyhedra.

Before proceeding, let us append some useful definitions.

For a given polyhedral partition $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i$, consider two subsets of polyhedral regions:

$$\{X\}_{I_N} = \{X\}_{I_L} \cup \{X\}_{I_C} \tag{6}$$

with an associated convex PWA lifting for $\{X\}_{I_L}$:

$$z_L(x) : \{X\}_{I_L} \to \mathbb{R} \tag{7}$$

Two polyhedral sets can be constructed:

$$P_L = \left\{ \begin{bmatrix} x \\ l \end{bmatrix} : x \in \mathcal{X}, l \geq \max\{z_{L_i}(x), i \in \mathcal{I}_L\} \right\} \tag{8}$$

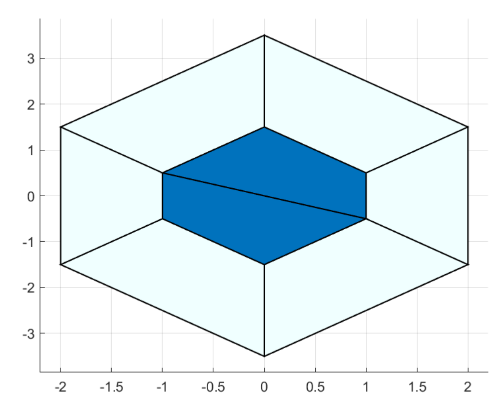


Fig. 6: Non-liftable subpartitions of the cell complex from Fig. 5 using the relaxation condition.

$$R_L = \mathrm{ConvHull}\left\{ \begin{bmatrix} v \\ z_L(v) \end{bmatrix} : v \in \mathcal{V}(\mathcal{X}_i), i \in \mathcal{I}_L \right\} \quad (9)$$

Definiton 5: A PWA convex lifting over $z : \mathcal{X} \to \mathbb{R}$ is *compatible* with the one over $\{X\}_{I_L}$ if

$$z(x) = z_L(x), \forall x \in \{X\}_{I_L}.$$

Theorem 1: A PWA convex lifting over $z : \mathcal{X} \to \mathbb{R}$ is *compatible* with $z_L : \{X\}_{I_L} \to \mathbb{R}$ if

$$P_L \supseteq \mathrm{Graph}\{z(.)\} \supseteq R_L. \quad (10)$$

Proof: For any point $x \in \{X\}_{I_L}$ there exist a corresponding point $[x^\top z_L(x)]^\top$ which belongs to the boundaries of both P_L and R_L. Consequently, the condition (10) indicates that necessarily $z(x) = z_L(x)$ and given the restriction on the argument, this corresponds to the definition of the compatibility of the lifting in between the partition and the subpartition. ■

Let us consider the example of the non-liftable partition $\{X\}_{I_N}$ illustrated in Fig. 5, with the previously identified non-liftable subpartition $\{X\}_{I_C}$ (dark blue in Fig. 6) and liftable subpartition $\{X\}_{I_L}$ (light blue in Fig. 6). The subpartition $\{X\}_{I_C}$ can be treated locally and a subdivision of $\{X\}_{I'_C}$ can be combined with $\{X\}_{I_L}$ if the lifting of $\{X\}_{I'_C}$ is compatible with the lifting of $\{X\}_{I_L}$.

The first challenge in constructing such a refinement is to ensure that exclusive refining of the non-liftable subpartition will ensure the liftability of the entire refined partition. In other words:

- Any new hyperplanes must not alter the outer boundaries of the subpartition $\{X\}_{I_C}$.
- Changes must not extend beyond the outer vertices of this subpartition.
- No modifications are allowed if they increase the complexity of the partition's cells.

Figure 7 illustrates this on the example in Fig. 5. In this context, the convex lifting with the convex hull R_L is the lower bound, whereas the lifting with extended facets on the outer boundaries of this convex hull P_L creates the lower bound for the lifting of a refined subpartition $\{X\}_{I'_C}$. Now, if such refinement $\{X\}_{I'_C}$ exists that its lifting is contained within this space shown as the red polytope in Fig. 7, while also maintaining continuity on the outer boundaries of $\{X\}_{I_C}$, then these two liftings can be seamlessly integrated by substituting the convex hull with the lifting of $\{X\}_{I'_C}$.

As long as $P_L \setminus R_L$ is not an empty set, there are infinitely many possible solutions to this problem, Fig. 8 showing two different choices. For this particular example the choices for refining $\{X\}_{I_C}$ have been selected by either using the projection of faces of its convex hull or by connecting all outer vertices to a single inner vertex. While these approaches may not automatically guarantee the liftability of the refined partition $\{X\}_{I'_C}$ in the general case, they can still provide partial solutions. This, in turn, allows for the local treatment of non-liftable subpartitions within $\{X\}_{I'_C}$, enabling the construction of the overall refinement of $\{X\}_{I_N}$ in a recursive manner.

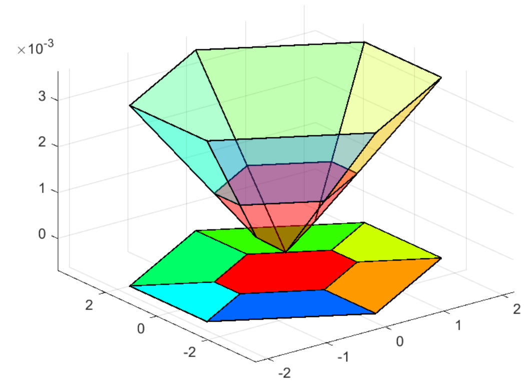
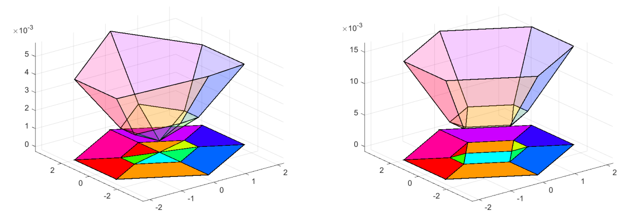
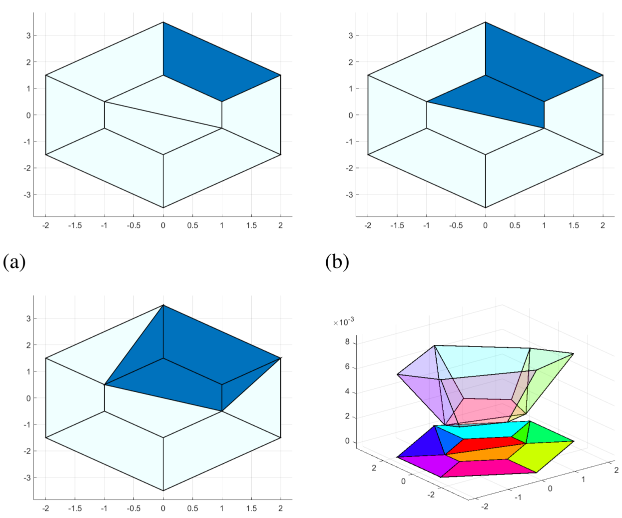


Fig. 7: Visualization of the space between the lower (R_L) and upper bound (P_L) for a compatible lifting of a subpartition.



(a) (b)

Fig. 8: Two examples of possible refinements of $\{X\}_{I_C}$ resulting in a compatible convex lifting.



(a) (b) (c) (d)

Fig. 9: Illustration of the procedure of extending and refining a non-liftable partition to achieve a compatible lifting. (a) Initial non-liftable partition identified via iterative building. (b) Extension to a larger non-liftable partition by adding a neighboring region. (c) Further extension to cover the convex hull of the non-liftable partition. (d) Final refined partition with convex lifting.

Although the example used to illustrate the above approach possesses a single, convex problematic subpartition, the caonstruction can be generalized. In case that the problematic subpartition is not convex, its extension to a convex subpartition shall be considered, e.g. by adding other neighboring polyhedra or extending it to cover its convex hull until a non-empty set $P_L \setminus R_L$ is created.

An example in this respect is shown in Fig. 9, where the initial non-liftable partition was previously identified via iterative construction as one single region (Fig. 9(a)), then extended to a larger non-liftable partition by adding neighboring region (Fig. 9(b)) and lastly, extended again, to cover its convex hull (Fig. 9(c)). In this case, it turns out that no additional refinement was needed, as shown in Fig. 9(d) depicting the refined partition and its convex lifting.

Furthermore, if multiple problematic subpartitions exist, the lifting of each, along with the larger modified lifting, can be adjusted or scaled to fulfill the convexity condition provided they can be accommodated within the available spaces while maintaining continuity.

VI. Conclusion

The paper presented a contribution to the structural treatment of convexly non-liftable polyhedral partitions or subpartitions. On one hand, it was shown that the convex liftability—a property that is essential for control design approaches relying on the construction of a convex lifting—can be enforced by subcuttings with a complexity lower than the one for the hyperplane arrangements. Such a global approach can be implemented iteratively and the quality of the refinement improved according to the tests on the hyperplane selection. On the other hand, treating the problem locally is feasible if certain conditions are met, allowing the preservation of precomputed elements from the existing solution, thus saving computational resources.

In general, for a local treatment of the convex non-liftability we suggest the following steps:

- Identify non-liftable subpartitions: Identifying local sources of non-liftability.
- Test if $P_L \setminus R_L = \emptyset$:
  - If empty: Extend the subpartition and test again.
  - If not empty: Find hyperplanes that can refine the subpartition in a way that allows for a compatible lifting.
- Repeat for each connected subpartition: Apply the above steps to every non-liftable subpartition within the partition.
- Incorporate refinements: Once all non-liftable subpartitions have been addressed, integrate the refinements into the original partition.

In refining the non-liftable subpartitions, any subdivision that maintains the cell complexity and does not extend beyond the subpartition being refined is feasible. Using projection of faces of the convex hull or connecting outer vertices to an inner vertex are recommended. However, restoring Delaunay triangulations or specific weighted Voronoi diagrams, as long as they meet the same conditions, are also viable options.

Additionally, extending non-liftable subpartitions requires a further discussion. It is important to clarify strategies such as prioritizing extensions based on neighboring subpartitions (graph or lattice structure), covering the convex hull, and exploring other structural solutions. These steps will increase the numerical robustness and improve the effectiveness.

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