Information Theoretic Sensor Placement Design for Optimal Filtering with Multirate Sampling

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***Abstract*— A typical chemical process involves measuring variables at different frequencies, resulting in a multirate system. In the current work, we propose a Kullback-Leibler divergence (KLD) based information theoretic approach for the sensor placement design problem with multirate sampling for a linear dynamical system. The proposed methodology is based on maximizing the quality of the estimates for a given measurement cost associated with sensor placement. The quality of estimation is defined using KLD which measures how far the estimation error density function for the given choice of sensors is from a user-specified estimation error density function. Thus, the proposed methodology incorporates user specified estimation accuracy directly in the design criteria. Multirate extension of traditional Kalman filter is used to quantify the estimation error density function. In particular, we propose an optimization problem where optimal sensor placement is the one for which the Kullback-Leibler (KL) based divergence of the estimation error density function (at large time instant) from the user specified reference density function is the minimum. Application on the benchmark quadruple tank and Tennessee Eastman case studies and comparison with existing design approaches demonstrates the utility of the proposed approach.**

I. Introduction and Background

State estimation is a pre-requisite for control. It involves estimating the internal states of a system, which are often not directly measurable, based on available sensor measurements and a mathematical model of the system. Sensor placement design (SPD) focuses on identifying the optimal measurements in a process. Careful choice of measurements plays a vital role in ensuring accurate state estimation, thereby enabling efficient monitoring and control. In traditional sensor placement design for linear dynamical systems, the problem is often formulated by optimizing (either minimizing or maximizing) a cost function derived from the Kalman filter's error covariance matrix of the estimates [1], [2].

The sensor placement problem for dynamical systems is typically categorized as either design time selection of sensors [1], [3], or as a time-varying sensor scheduling problem [4], accounting for changes in sensor availability. There is literature on design of multirate control systems [5], [6], however, the literature for sensor placement design in multirate system is relatively limited. Multirate system involve sampling of different measurements at different rates [7]. Such scenarios often arise in process systems, where secondary variables such as temperature, pressure etc., are measured at higher frequency while some key variables such as related to concentration and quality are measured less frequently [8]. In the context of sensor placement design, Mehra [9] presented optimization of measurement schedules and sensor placement design by considering the state prediction error covariance matrix for linear dynamic systems. Kadu et al. [10] presented sensor network design for multirate linear dynamical systems. They also considered error covariance matrix of the estimates and proposed the problem of sensor network design in multirate settings that considered the sensors cost and sampling frequencies to assess the trade-off between quality of the estimates and the overall cost of the measurement system. The various sensor placement design approaches in the literature optimize error covariance matrices through minimization or maximization of associated measures, such as A-, D-, E-, and T-optimality [1], [11], [12]. These approaches implicitly assume that the estimates have a Gaussian density since they work with only covariance matrices. Further, there is no provision for the end-user to specify her requirements about the performance in the sensor placement design procedure. Recently, Patel and Bhushan [3] presented a Kullback-Leibler Divergence (KLD) based sensor placement design for optimal filtering in linear dynamical systems using Kalman filter. The key idea in their work was to minimize the KLD measure between a user-specified reference density function of error estimates and the density function obtained by the selected sensors. However, their work was limited to single rate processes.

In this work, we propose a sensor placement design for multirate systems that uses the KLD to optimize Kalman filter based state estimation. Utilizing KLD, the aim is to select variables to measure such that the estimated state density function closely matches a user-defined reference density function, subject to some design constraints. This approach allows direct integration of user preferences into the design process. Further, since the KLD works with the probability densities, it is applicable even to non-Gaussian estimation problems. However, in this study, we focus only on cases where the estimates follow a Gaussian distribution.

II. Problem Statement

Consider a discrete linear time invariant system with time varying measurements of the form:

$$\mathbf{x}_{k+1} = \Phi \mathbf{x}_k + \Gamma u_k + \mathbf{w}_k \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{v}_k \tag{2}$$

where, k indicates k^{th} time instant, $\mathbf{x}_k \in \mathbb{R}^n$ is the system state, $\Phi \in \mathbb{R}^{n\times n}$ is the system dynamics matrix, and $\mathbf{w}_k \in \mathbb{R}^n$

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is a zero mean white Gaussian noise process with covariance $\mathbf{Q}$. Let $\mathbf{C} \in \mathbb{R}^{q\times n}$ denote the full measurement matrix corresponding to all the available sensors and $\mathbf{R}$ denotes the covariance matrix of the zero-mean white Gaussian noise corresponding to the measurement matrix $\mathbf{C}$. Given a multirate sampling scenario, where the sampling rate of each measurement may be different (but known), $\mathbf{C}_k \in \mathbb{R}^{q_k\times n}$ denotes the measurement matrix corresponding to measurements available at time instant k. $\mathbf{v}_k \in \mathbb{R}^{q_k}$ is the corresponding zero-mean white Gaussian measurement noise with covariance $\mathbf{R}_k$. We assume that the process noise and the measurement noise are independent of each other. Also, they are assumed to be independent of the initial state $\mathbf{x}(0)$ which is assumed to be a Gaussian random variable as: $\mathbf{x}(0) \sim \mathcal{N}(\hat{\mathbf{x}}_{0|0}, \mathbf{P}_{0|0})$.

The discretization interval for the above model is T sec. The sampling interval of i^{th} sensor is τ_i and is specified. It is assumed that τ_i is an integer multiple of T with $s_i = \tau_i/T$. Henceforth, with some abuse of notation, we refer to s_i as the 'sampling period' of the i^{th} sensor. For $k = 1,\ldots,N$, if the remainder of $k/s_i = 0$, then measurement from sensor i is available at time k. The matrices $\mathbf{C}_k, \mathbf{R}_k$ can be constructed from matrices $\mathbf{C}$, and $\mathbf{R}$, respectively, by defining the following indicator variable:

$$d_{ik} = \begin{cases} 1, & \text{if remainder } \{k/s_i\} = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Here, $d_{ik} = 1$ indicates availability of i^{th} sensor measurement at time instant k. The matrices $\mathbf{C}_k, \mathbf{R}_k$ at the k^{th} sampling instant are obtained as:

$$\mathbf{C}_k = \text{defl}\{D_k \mathbf{C}\} \tag{4a}$$

$$\mathbf{R}_k = \text{defl}\{D_k \mathbf{R} D_k\} \tag{4b}$$

$$\text{where, } D_k = diag[d_{1k}, d_{2k}, \ldots, d_{qk}]$$

where, defl$\{.\}$ operator which operates on a matrix denotes removal of all zero rows from its argument matrix $\{.\}$.

For purpose of sensor placement design, let set $\mathcal{Q}$ consist of the q available sensors with the i^{th} sensor corresponding to the i^{th} row of $\mathbf{C}$ matrix. Let $\widetilde{\mathbf{C}}$ be a matrix obtained from $\mathbf{C}$ by stacking rows of $\mathbf{C}$ corresponding to the selected sensors. A given choice of sensor placement results in corresponding measurement matrix $\widetilde{\mathbf{C}}_k$ and measurement noise covariance matrix $\widetilde{\mathbf{R}}_k$ at each k. In particular, $\widetilde{\mathbf{C}}_k$ is obtained from $\mathbf{C}_k$ by stacking rows of $\mathbf{C}_k$ corresponding to the selected sensors. Similarly, $\widetilde{\mathbf{R}}_k$ is obtained from $\mathbf{R}_k$ by extracting blocks from $\mathbf{R}_k$ corresponding to the selected sensors.

The problem considered in this work is to place sensors in a multirate setting given an upper limit on the resources or available cost. Each sensor $i \in \mathcal{Q}$ has an associated measurement cost $c_i \in \mathbb{R}_{\geq 0}$. In general, measurement cost includes capital cost (purchase/installation cost) of the sensors, as well as operating (sampling cost) of the sensors. In the current work, we assume that all the sensors are already installed and hence ignore the capital cost. Thus, in a multirate setting, if sensor i is available at time instant k, sampling cost of i is incurred only at that time instant k. This setting allows us to investigate effect of sampling rates on the sensor placement results. The SPD problem in this work involves selecting optimal measurements for state estimation in a multirate setting, constrained by a total sampling cost C^*. The goal is to allocate the sampling budget efficiently based on given sampling rates.

III. Preliminaries

In this section we discuss some definitions and relevant background material.

A. Kalman Filter for multirate systems

The Kalman filter is a recursive estimation approach that involves two primary steps: (i) the prediction step, and (ii) the update step. Let $\hat{\mathbf{x}}_{k-1|k-1}$ and $\mathbf{P}_{k-1|k-1}$ denote, respectively, the mean and covariance of the conditional Gaussian state distribution at time instant $k-1$, after processing measurements up to time instant $k-1$. Below, we outline the prediction and update steps to compute the corresponding state estimates and covariance at time instant k.

Prediction Step: The conditional state density evolves according to the discrete LTI of the form (1) through propagation of its mean and covariance:

$$\hat{\mathbf{x}}_{k|k-1} = \Phi \hat{\mathbf{x}}_{k-1|k-1} + \Gamma \mathbf{u}_k \tag{5}$$

$$\mathbf{P}_{k|k-1} = \Phi \mathbf{P}_{k-1|k-1} \Phi^T + \mathbf{Q} \tag{6}$$

where $\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}$ are the mean and covariance of predicted conditional state Gaussian density.

Update Step: When measurements of the system states become available at the k^{th} time instant, the Kalman filter updates both the state estimate mean and covariance matrix through the measurement update step:

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{L}_k(\mathbf{y}_k - \mathbf{C}_k \hat{\mathbf{x}}_{k|k-1}) \tag{7}$$

$$\mathbf{P}_{k|k} = [\mathbf{I} - \mathbf{L}_k \mathbf{C}_k] \mathbf{P}_{k|k-1} \tag{8}$$

where, $\mathbf{L}_k = \mathbf{P}_{k|k-1} \mathbf{C}_k^T [\mathbf{C}_k \mathbf{P}_{k|k-1} \mathbf{C}_k^T + \mathbf{R}_k]^{-1}$ is the Kalman gain. In equations (7), (8), $\hat{\mathbf{x}}_{k|k}$ and $\mathbf{P}_{k|k}$ are the mean and covariance, respectively, of the conditional state Gaussian density conditioned on measurements till time instant k. Further, $\hat{\mathbf{x}}_{k|k}$ is the optimal estimate of the states [13]. The estimation error at k^{th} time instant is defined as:

$$\varepsilon_{k|k} \triangleq \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \tag{9}$$

where $\mathbf{x}_k$ is the true (unknown) state at time k. The estimation error then has a Gaussian density as $\varepsilon_{k|k} \sim \mathcal{N}(0, \mathbf{P}_{k|k})$ [13].

B. Kullback-Leibler divergence

The Kullback-Leibler divergence (KLD) is the measure of information discrepancy between a density function $g(\mathbf{x})$ from a reference density function $f(\mathbf{x})$ [14], where, $\mathbf{x} \in \mathbb{R}^n$ be a continuous random variable, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are the pdfs of $\mathbf{x}$, such that $g(\mathbf{x}) = 0 \implies f(\mathbf{x}) = 0,\ \forall \mathbf{x} \in \mathbb{R}^n$. Then KLD (denoted by D_{KL}) is defined as [15]:

$$D_{KL}(f(\mathbf{x})||g(\mathbf{x})) := \int_{-\infty}^{\infty} f(\mathbf{x}) \ln\left(\frac{f(\mathbf{x})}{g(\mathbf{x})}\right) d\mathbf{x} \tag{10}$$

KLD satisfies the following properties:

1) $D_{KL}(f(\mathbf{x})||g(\mathbf{x})) \geq 0$, for all $f(\mathbf{x})$ and $g(\mathbf{x})$
2) $D_{KL}(f(\mathbf{x})||g(\mathbf{x})) = 0$, if and only if $f(\mathbf{x}) = g(\mathbf{x})$

For the special case when $\mathbf{x}$ is Gaussian with $f(\mathbf{x}) = \mathscr{N}(\mu_f, \Sigma_f)$, and $g(\mathbf{x}) = \mathscr{N}(\mu_g, \Sigma_g)$, KLD can be analytically obtained as:

$$D_{KL}(f(\mathbf{x})||g(\mathbf{x})) = \frac{1}{2}\left[tr\left(\Sigma_f \Sigma_g^{-1}\right) + \ln\left(\frac{det(\Sigma_g)}{det(\Sigma_f)}\right) + (\mu_g - \mu_f)^\top \Sigma_g^{-1}(\mu_g - \mu_f) - n\right] \quad (11)$$

where, $tr(\cdot)$ and $det(\cdot)$ denote the trace and determinant of a matrix $(\cdot)$, respectively.

C. Optimal design criteria

In the literature, several widely-used alphabetical optimality criteria [11] exist, including:

(a) **A-optimal design**- minimizes the trace of the error covariance matrix of the estimates [10][1];
(b) **D-optimal design**- minimizes the determinant of the error covariance matrix of the estimates [11];
(c) **E-optimal design**- maximizes the minimum eigenvalue of the inverse of error covariance matrix of the estimates [12][16];
(d) **T-optimal design**- maximizes the trace of the inverse of error covariance matrix of the estimates [12].

IV. Proposed Methodology

In the current work, we present a sensor placement design methodology for linear dynamical systems operating in multirate settings, based on the Kullback-Leibler (KL) divergence. The proposed approach minimizes the KL divergence between the state estimation error density and a user-defined reference density, utilizing Kalman filtering for state estimation for the selected measurements. As established in Section III-A, the estimation error follows a Gaussian distribution. While the error mean remains zero, its covariance matrix $\mathbf{P}_{k|k}$ evolves with time. For the multirate sampling scenario, due to change of available measurements with time, the error covariance matrix does not converge to a single steady state and instead a periodic pattern for the state error covariance matrix is obtained. Hence, for the sensor placement design for multirate systems, it is more realistic to take the dynamic evolution of the covariance matrix over a sufficiently long duration into consideration [10]. Thus, the simulation interval for multirate Kalman filtering should be chosen large enough to allow the state error covariance matrix to attain steady values for various possible sensor network combinations [10]. Based on this idea, an SPD formulation for multirate linear dynamical systems is proposed next.

A. KLD based sensor placement design

In this work, we propose the following SPD approach:
Formulation $\mathscr{P}_1$

$$\min_{z} \quad \widetilde{D}_{KL} \quad (12)$$

$$\text{s.t.} \quad \sum_{k=1}^{N}\sum_{i=1}^{q} d_{ik}c_i z_i \leq C^*, \quad (13)$$

$$\mathbf{z} \in \{0,1\}^q, \text{and} \quad (14)$$

$$\text{system is detectable} \quad (15)$$

Formulation $\mathscr{P}_1$ is discussed next:
1) Objective function: The objective function is defined as:

$$\widetilde{D}_{KL} = \frac{1}{N}\sum_{k=1}^{N} w_k D_{KL}(f(\varepsilon_{k|k})||g(\varepsilon_{k|k})) \quad (16)$$

It involves weighted average of KLD values at N time instants with $w_k > 0,\ k = 1,2,...,N$ and $\sum w_k = N$ being the user specified weights. For example, choosing w_k to be relatively higher for large k ensures that the user specifications are met more closely at later time instants than at initial time instants for the optimal sensor placement. $\varepsilon_{k|k}$ is the state estimation error at time instant k (9). At the k^{th} time instant, the reference pdf $f(\varepsilon_{k|k})$ of the state estimation error is taken to be 0-mean Gaussian density, i.e. $f(\varepsilon_{k|k}) = \mathscr{N}(0, \mathbf{P}_{k|k}^{ref})$, where the error covariance matrix $\mathbf{P}_{k|k}^{ref}$ is specified by the user. The design pdf at time instant k, i.e. $g(\varepsilon_{k|k})$ is taken to be the estimation error density function obtained at k^{th} time instant using a multirate Kalman filter for the designed sensor placement. This density is also 0-mean Gaussian density i.e. $g(\varepsilon_{k|k}) = \mathscr{N}(0, \mathbf{P}_{k|k})$ where the covariance matrix of the estimates, $\mathbf{P}_{k|k}$ is the error covariance matrix (8) for the selected sensors.
2) Cost constraint: In (13) the term in the LHS represents the total measurement cost incurred over N instants for a given choice of sensors. Here, c_i represents the operating cost of the i^{th} sensor, the variable d_{ik} is either 0 or 1 as defined in (3), and z_i indicates whether the i^{th} sensor is selected or not. This constraint ensures that the total cost of selected sensors is not more than the allowed cost C^*.
3) Binary constraints: In (14), q is the number of sensors available for sensor placement. The decision variable vector $\mathbf{z}$ is then defined as:

$$z_i = \begin{cases} 1, & \text{if } i^{th} \text{ sensor is selected} \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

It is to be noted that in the current work only one sensor for each measurement is available. This is for simplicity and the design approach can be easily extended to situations where hardware redundancy is allowed.
4) Detectability constraint: Equation (15) is a constraint ensuring system detectability as an additional requirement imposed for sensor placement. A system is said to be detectable if the unstable modes of Φ are observable [17]. A single-rate system is observable if the rank of the observability matrix equals the number of states. For a multirate system, defining

the observability criteria is not straightforward. Wang et al.[7] provided the following condition for a multirate single-input, single-output (SISO) system:

Let the input and output be available at periods $b\delta$ and $c\delta$, respectively, where b and c are positive integers, and δ is the base sampling period. Using the lifting technique, the multirate system is associated with an equivalent linear time-invariant (LTI) state-space model, from which a fast-rate discrete-time system is extracted. For every eigenvalue λ of Φ, if none of the $(bc-1)$ points

$$\lambda e^{j\frac{2\pi l}{bc}}, \quad l = 1,2,\dots,bc-1 \tag{18}$$

are eigenvalues of Φ, then the lifted multirate system is detectable if the fast-rate model is detectable [7]. For a multi-input, multi-output (MIMO) case, b and c are considered as the least common multiples of all the input and output periods, respectively [10]. The detectability of the fast-rate system can be checked based on matrices Φ and $\widetilde{\mathbf{C}}$, where $\widetilde{\mathbf{C}}$ is the measurement matrix corresponding to the chosen sensors. If the fast-rate system is detectable, and Φ satisfies the condition given in (18), then the multirate system is also detectable [10].

The formulation $\mathscr{P}_1$ is an integer nonlinear programming problem. The computation of D_{KL} is explained next.

B. Computation of D_{KL}

In this section we present the computation of D_{KL} in $\mathscr{P}_1$. Given the discrete linear time invariant (LTI) system of the form (1), sampling rate s_i for each measurement, and a set of selected sensors, matrices $\widetilde{\mathbf{C}}_k$ and $\widetilde{\mathbf{R}}_k$ are constructed as discussed in Section II. Then the error covariance matrix $\mathbf{P}_{k|k}$ of the Gaussian estimation errors is obtained using (8) of the multirate Kalman Filter. Further, given the error covariance matrix $\mathbf{P}^{ref}_{k|k}$ of the estimation errors specified by the end-user, the KLD reduces to:

$$D_{KL}(f(\varepsilon_{k|k})||g(\varepsilon_{k|k})) = \frac{1}{2}\Bigg[tr(\mathbf{P}^{ref}_{k|k}\mathbf{P}^{-1}_{k|k}) + \ln\left(\frac{det(\mathbf{P}_{k|k})}{det(\mathbf{P}^{ref}_{k|k})}\right) - n\Bigg] \tag{19}$$

where, n is the number of states. KLD thus computed is used in (12) to compute the SPD objective.

In the next section, we present two case studies: (i) quadruple tank system, and (ii) Tennessee Eastman (TE) process. Although these systems are nonlinear, in the current work we restrict to the linearized (around an operating point) versions of these systems. For both these linearized systems, the resulting Φ matrices are stable, and hence the detectability criterion is satisfied irrespective of the sensor placement. The solution of Formulation $\mathscr{P}_1$ is obtained by enumeration for the quadruple tank problem and by Genetic Algorithm (GA) for the TE problem.

V. Case Study 1: Quadruple Tank

We consider a quadruple tank system from literature [18]. The system consists of four states and the system dynamics are governed by the following equations:

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \tag{20a}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \tag{20b}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 \tag{20c}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 \tag{20d}$$

where A_i is the cross-section of Tank i; a_i is the cross-section of the outlet hole; h_i is the water level; $\mathbf{v}_i$ is the voltage applied to Pump i and the corresponding flow is $k_i v_i$, for $i = 1,2,\dots,4$. The acceleration due to gravity is denoted by g. The model is linearized (using Taylor series expansion) around the operating point [18] listed in Table I and then discretized with sampling interval of 5 secs to obtain a linear, discrete-time dynamic model. For this problem, the number of states, $n = 4$ and the number of measurements, $q = 4$. The Φ matrix and other matrices considered for SPD are:

TABLE I: Parameter values and the chosen operating point for quadruple tank [18]

State/ Parameters	Values	Unit	State/ Parameters	Values	Unit
(h_1,h_2)	$(12.4,12.7)$	[cm]	A_1,A_3	28	$[\mathrm{cm}^2]$
(h_3,h_4)	$(1.8,1.4)$	[cm]	A_2,A_4	32	$[\mathrm{cm}^2]$
(v_1,v_2)	$(3.00,3.00)$	[V]	a_1,a_3	0.071	$[\mathrm{cm}^2]$
(k_1,k_2)	$(3.33,3.35)$	$[\mathrm{cm}^3/\mathrm{Vs}]$	a_2,a_4	0.057	$[\mathrm{cm}^2]$
(γ_1,γ_2)	$(0.70,0.60)$		g	981	$[\mathrm{cm/s}^2]$

$$\Phi = \begin{bmatrix} 0.9233 & 0 & 0.1813 & 0 \\ 0 & 0.9462 & 0 & 0.1493 \\ 0 & 0 & 0.8112 & 0 \\ 0 & 0 & 0 & 0.8465 \end{bmatrix} \tag{21}$$

$$\mathbf{C} = I_{4\times4}, \quad \mathbf{Q} = I_{4\times4}, \quad \mathbf{R} = I_{4\times4} \tag{22}$$

In this example case study, we assume that there is one-to-one correspondence between the sensors and the states and hence, $\mathbf{C}$ matrix is diagonal ($I_{4\times4}$). The costs of all sensors is taken to be 100 units. The sampling rates (s_i, $i = 1,2\dots4$) for measurements h_1, h_2, h_3, and h_4 are 20, 2, 1, and 10, respectively. We consider $N = 500$ instants as the simulation duration and perform SPD for the case when reference error covariance matrix, $\mathbf{P}^{ref}_{k|k} = I_{4\times4}$ $\forall k = 1,2,...,N$. Table II lists the sensor placement designs obtained by solving formulation $\mathscr{P}_1$ for various values of C^*. As evident from Table II, the KL divergence metric $\widetilde{D}_{KL}$ exhibits a monotonic decrease with increasing measurement cost budget C^*. This behavior is expected since the initial state estimation quality is suboptimal, characterized by significant divergence between the estimated density and the user-specified reference distribution and progressive allocation of measurement (increase in C^*) leads to improved estimation accuracy, thereby reducing the KL divergence Table III compares the proposed approach with conventional optimality criteria

TABLE II: Optimal SPD for quadruple tank

C^*	Cost utilised	$\widetilde{D}_{KL}$	States selected for measurements
10000	7500	1.63	$\{h_1, h_4\}$
30000	27500	0.98	$\{h_1, h_2\}$
60000	32500	0.90	$\{h_1, h_2, h_4\}$
90000	84200	0.68	$\{h_1, h_2, h_3, h_4\}$

for sensor placement design. The table shows the optimal sensor placement obtained using A-, D-, E-, and T-optimality criteria and the corresponding $\widetilde{D}_{KL}$ values between the design error density function and the reference density function The results show that all criteria except T-optimality yield identical sensor placement designs (SPD) as presented in Table III. Figure 1 shows the values of time varying D_{KL} and time varying $T-$optimality values for $C^* = 30000$. The resulting sensor placement solving $T-$ optimality is $\{h_2, h_4\}$, which is different than the optimal sensor placement resulting from solving $\mathscr{P}_1$. Figure 1 shows three profiles, where,

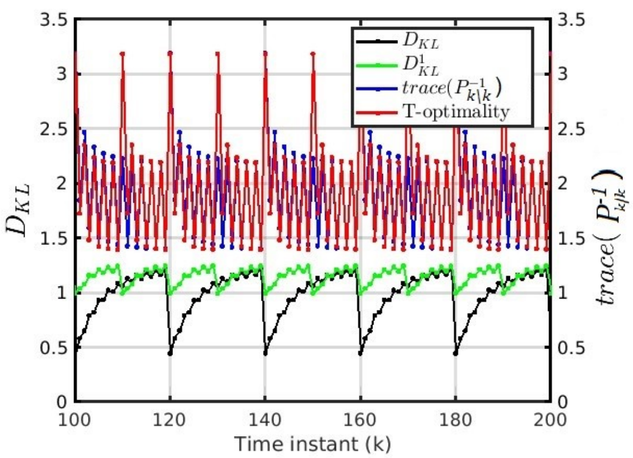

Fig. 1: Values of D_{KL} and $trace([\mathbf{P}_{k|k}]^{-1})$ ($T-$optimality criterion) shown for $k = 100$ to 200 instants for $C^* = 30000$

black profile shows the time varying D_{KL} for optimal sensor placement design obtained by solving $\mathscr{P}_1$, blue profile is for $trace(\mathbf{P}^{-1}_{k|k})$ values for sensor placement corresponding to optimal design resulted from $\mathscr{P}_1$, red profile corresponds to the optimal sensor placement design resulting from maximizing $trace(\mathbf{P}^{-1}_{k|k})$ ($T-$ optimality), and green profile (D^1_{KL}) shows the KLD values corresponding to the optimal sensor placement resulting by solving $T-$optimality criteria. For clarity, the figure is shown only for time instants 100 to 200. The black profile shows a slight decrease every 2 instants as h_2 becomes available, and reaches a minimum value every 20 instants when both h_1 and h_2 are available, representing the optimal case. The blue profile indicates that the availability of h_1 and h_2 corresponds to the maximum value of $trace(\mathbf{P}^{-1}_{k|k})$. The red profile represents the maximum value of $trace(\mathbf{P}^{-1}_{k|k})$ obtained using the $T-$optimality criteria. The corresponding D_{KL} values (green profile) reach minima every 20 instants when h_2 and h_4 are available. However, this result is suboptimal compared to the solution of $\mathscr{P}_1$.

TABLE III: $\widetilde{D}_{KL}$ v/s $A-$, $D-$, $E-$ and $T-$ optimality values for various Sensor placement results

C^*	$A-$	$D-$	$E-$	$T-$
10000	24.51 ($\widetilde{D}_{KL} = 1.63$)	598.86 ($\widetilde{D}_{KL} = 1.63$)	0.085 ($\widetilde{D}_{KL} = 1.63$)	1.18 ($\widetilde{D}_{KL} = 1.63$)
30000	14.51 ($\widetilde{D}_{KL} = 0.98$)	74.14 ($\widetilde{D}_{KL} = 0.98$)	0.15 ($\widetilde{D}_{KL} = 0.98$)	1.96 ($\widetilde{\mathbf{D}}_{\mathbf{KL}} = \mathbf{1.13}$)
60000	13.79 ($\widetilde{D}_{KL} = 0.90$)	61.81 ($\widetilde{D}_{KL} = 0.90$)	0.15 ($\widetilde{D}_{KL} = 0.90$)	2.55 ($\widetilde{\mathbf{D}}_{\mathbf{KL}} = \mathbf{1.41}$)
90000	9.26 ($\widetilde{D}_{KL} = 0.68$)	9.36 ($\widetilde{D}_{KL} = 0.68$)	0.26 ($\widetilde{D}_{KL} = 0.68$)	3.47 ($\widetilde{D}_{KL} = 0.68$)

VI. CASE STUDY 2: TENNESSEE EASTMAN (TE)

We consider Tennessee Eastman problem from literature [19]. The system consists of twenty six states and its dynamic model is given by a set of nonlinear differential equations. The dynamic nonlinear model is linearized at steady state values given in Table IV to obtain the corresponding linear, time-invariant system in continuous-time. The resulting system is then discretized with sampling interval of 0.05 sec, to obtain the corresponding discrete time linear time invariant system. The computational time using enumeration

TABLE IV: Steady-state values of the state variables [10]

State	Varia -ble	Cost	Steady state value	State	Varia -ble	Cost	Steady state value
N_{Ar}	x_1	835	5.12	N_{Fm}	x_{14}	1757	29.53
N_{Br}	x_2	1441	3.36	N_{Gm}	x_{15}	55	19.39
N_{Cr}	x_3	10	2.06	N_{Hm}	x_{16}	1341	11.87
N_{Dr}	x_4	605	0.13	N_{As}	x_{17}	835	0.06
N_{Er}	x_5	294	9.15	N_{Bs}	x_{18}	1118	4.51
N_{Fr}	x_6	185	2.99	N_{Cs}	x_{19}	281	1.47
N_{Gr}	x_7	373	67.31	N_{Ds}	x_{20}	397	20.84
N_{Hr}	x_8	692	66.01	N_{Es}	x_{21}	1602	16.46
N_{Am}	x_9	794	56.72	N_{Fs}	x_{22}	1937	29.41
N_{Bm}	x_{10}	1078	26.01	N_{Gs}	x_{23}	627	7.03
N_{Cm}	x_{11}	839	33.02	N_{Hs}	x_{24}	1385	8.20
N_{Dm}	x_{12}	1371	10.63	N_{Gp}	x_{25}	1753	3.96
N_{Em}	x_{13}	409	21.74	N_{Hp}	x_{26}	1790	17.73

of all feasible combinations (2^{26} combinations) of sensor placement design is high for this case study and hence, genetic algorithm (GA) is used to solve $\mathscr{P}_1$. Parameters used for GA and multirate Kalman filter are given in Table V.

TABLE V: Genetic Algorithm (GA) and multirate Kalman filter parameters for TE process

Parameter	**Value**
Population size	100
Number of generations	200
Crossover fraction	0.8
Constraint Tolerance	$1e^{-3}$
N for Kalman filter	500
$\mathbf{Q}$	$10^{-6}\times$ Identity matrix
$\mathbf{R}$	Identity matrix

Unlike case study 1 where we restricted the weights to be equal over the time, in this case study, we perform a sensitivity analysis by varying weights w_k. The reference error covariance for this case corresponds to the steady state solution of Ricatti equation when all the variable are available for measurement at all time instant i.e., $\mathbf{P}^{ref}$ is constant. Table IV lists the randomly chosen measurement wise cost. The sampling rates $(s_i, i = 1, 2, ..., 26)$ for the available measurements are $[10, 10, 6, 10, 5, 8, 3, 4, 2, 5, 2, 4, 7, 8, 5, 2, 1, 1, 6, 9, 5, 9, 8, 4, 2, 2]$ which are randomly chosen. Problem $\mathscr{P}_1$ is solved for

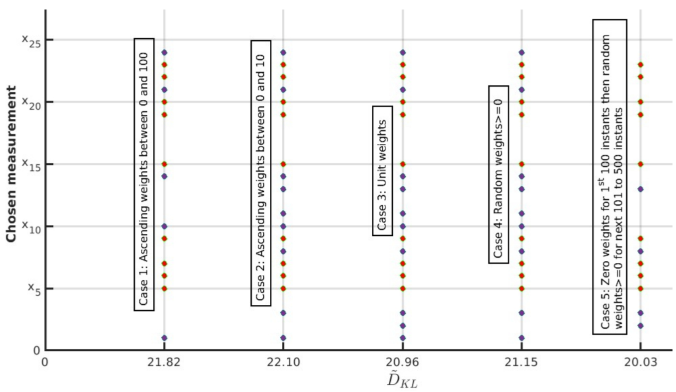

Fig. 2: Optimal sensor placement and $\widetilde{D}_{KL}$ for different weights for $C^* = 300000$

different combinations of w_k and the results are presented in Fig. 2 for 5 cases: Case 1: w_k are ascending weights between 0 and 100, Case 2: w_k are ascending weights between 0 and 10, Case 3: w_k are unit weights, Case 4: w_k are random weights ≥ 0, and Case 5: w_k are zero weights for first 100 instants and then w_k are the random weights ≥ 0 for next $k = 101 - N$ instants. The analysis shows that for $C^* = 300000$, variables $\{x_5, x_6, x_7, x_{15}, x_{19}, x_{20}, x_{22}, x_{23}\}$ are always in set of chosen measurement because the cost of measurement is low and sampling rate is relatively low for these measurements. As for variable $\{x_{25}, x_{26}\}$, one can see that the cost of measurement is high and sampling rate is also relatively higher for these measurements. Thus, these measurements are never selected for the considered cases.

VII. Conclusions

This work develops an information-theoretic framework for optimal sensor placement in linear dynamical systems with multirate sampling. The proposed methodology explicitly incorporates user preferences through a reference estimation error distribution and formulates the design problem as minimization of the weighted Kullback-Leibler (D_{KL}) divergence between the actual and desired error distributions Numerical case studies on both quadruple tank and Tennessee Eastman processes demonstrate that the D_{KL}-based approach yields sensor placement design that better approximate the target distribution and performs better than the conventional optimality criteria which do not consider user-specified performance in their optimization problem. In the current work, only Gaussian density functions were considered. In future, application of the proposed approach for non-Gaussian estimates can be investigated.

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