Comparison of recursive and nonrecursive processing schemes in the federated filtering

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Abstract—**A linear estimation problem of decentralized processing of measurements is considered, which involves the estimate of the state vector of a dynamic system by fusion of the estimates generated in local filters using the data from distributed (separate) sensors. The conditions are discussed that ensure the coincidence of the estimates and calculated covariance matrices of estimation errors generated in this way with the results obtained in the optimal centralized Kalman filter. The features of these conditions for recursive and nonrecursive schemes are analyzed. The comparison of two schemes is illustrated using an example of random walk estimation.**

I. INTRODUCTION

It is known that in the case of sensor redundancy, the linear estimation problem can be solved on the basis of two variants: the centralized one, wherein all available measurements are processed in a single filter, and the decentralized variant, which assumes the presence of a set of separate (local) sensors and its essence is that initial processing is implemented in local filters (LF) that generate local estimates. The final (global) estimate is calculated by fusion of local estimates in the master filter. In integrated navigation systems with a distributed structure of sensors, which are commonly called a local system, the decentralized variant is commonly implemented using federated filtering algorithms (FFA).

FFAs have recently been widely used to solve estimation problems based on data from distributed sensors [3-11], which is relevant, in particular, for AUV group navigation [4,5], indoor navigation [5, 6], and integrated navigation systems [7,8]. The main advantages of the FFAs usually include a lower computational load compared to the centralized Kalman filter and their immunity to false sensor measurements [1,2]. It should be noted that FFAs are commonly designed in relation to recursive processing schemes using sequential processing of each measurement. As a consequence, two types of FFAs are usually distinguished. One of them is the FFA without reset of LFs, in which the obtained global estimate is not involved in the calculation of local estimates. The other one is FFAs with reset of LFs, in which the estimates generated in the master filter are taken into account in the calculation of local estimates [1-11]. The features of recursive FFAs based on the so-called information-sharing principles and their application to navigation data processing are considered in detail in [1].

Further development of such algorithms is discussed in [2], in which the authors show the possibility of ensuring the guaranteeing properties of FFA, meaning the fact that the real covariance matrix of the global estimate does not exceed the calculated covariance matrix.

The main disadvantage of FFA is that, in the general case, these algorithms do not ensure optimal estimates and the covariance matrices calculated in the master filter is not the real covariance matrix. To reduce errors of the global estimate, different adaptive methods for calculation of the information sharing factors are proposed [3,4,10,11]. For some particular cases, the conditions of information sharing are formulated for the so-called consistent tuning of LFs, which allows obtaining an optimal estimate in the master filter [2,7]. However, the authors of this paper are not aware of any general answer to the question about the conditions ensuring the coincidence of the estimates and the calculated covariance matrices generated in the master filter with the results obtained in the optimal centralized Kalman filter.

As mentioned above, FFAs are recursive with respect to measurements. At the same time, the last decade has seen an increased interest in nonrecursive processing methods, including algorithms based on factor-graph optimization [12-14]. A number of papers show the advantage of nonrecursive schemes as compared to traditional recursive algorithms, for example, the Kalman filter [12,13]. However, to date, FFAs have been studied only in relation to traditional recursive schemes, but the features of nonrecursive FFAs have not been considered in the literature yet. As shown below, it is the change-over to a nonrecursive processing scheme that allows us to answer the question formulated above.

In this regard, the aim of the proposed work is to study the FFAs features in the implementation of nonrecursive processing schemes.

The paper is structured as follows. In Section 2, we formulate the problem statement and consider the recursive scheme for calculating the optimal estimate. Section 3 provides a brief overview of the features of recursive FFAs. We also discuss the conditions of consistent tuning. Section 4

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proposes a solution to the problem of calculating the optimal estimate and estimation using the federated filter applied to a nonrecursive scheme. Conditions for FFAs consistent tuning are described. Section 5 provides an example illustrating the comparison of the recursive and nonrecursive FFAs.

II. A PROBLEM STATEMENT. RECURSIVE SCHEME FOR CALCULATION OF THE OPTIMAL ESTIMATE

Let us formulate the estimation problem under study. Assume that we have an *n*-dimensional Gaussian Markov sequence

$$x_k = \Phi_k x_{k-1} + \Gamma_k w_k, x_0 \in N\{x_0; \bar{x}, P_0\}, w_k \in N\{w_k; 0, Q\}, \quad (1)$$

and the following set of vector measurements

$$y_k^j = h_k^j x_k + v_k^j; \quad v_k^j \in N\{v_k^j; 0, R^j\}, j = 1...m, \quad (2)$$

where k is a discrete moment of time, Φ_k and Γ are known $n \times n$ - and $n \times p$ - dimensional matrices; y_k^j are local *l*-dimensional *j*-th vector measurements; w_k is the Gaussian zero-mean *p*-dimensional vector of system noise with known covariance matrices Q; v_k^j are Gaussian zero-mean *l*-dimensional vectors of measurement noise with covariance matrices R^j; h_k^j are the known $l \times n$ dimensional measurement matrices; v_k^j, w_k are assumed to be white-noise sequences, independent of each other and of x_0. For the sake of simplicity, without loss of generality, the dimensions of all *j*-th measurements y_k^j, $j = 1...m$ are assumed to be the same and equal to *l*. From here on, $a \in N\{a; \bar{a}, A\}$ is used for the Gaussian random vector a with mathematical expectation $\bar{a}$ and covariance matrix A.

The problem is to determine the optimal—in the mean-square sense—estimate of vector x_k at the moment of time *k* using measurements (2), accumulated by the current *k*-th moment of time, and the corresponding covariance matrix P_k of the estimation errors.

It is important to note the following feature of the problem under consideration: a set of *m* measurements generated by separate local sensors is available at each moment of time *k*.

If we introduce a composite lm -dimensional vector $y_k^C = \left(y_k^1, y_2^2 ... y_k^m\right)^T$, then measurements (2) can be written as

$$y_k^C = h_k^C x_k + v_k^C, \ v_k^C \in N\{v_k^C; 0, R^C\}, \quad (3)$$

where $v_k^C = \left(v_k^1, v_k^2 ... v_k^m\right)^T$ is an lm -dimensional vector, $R^C = diag\left(R^1,..R^m\right)$ is an lm -dimensional matrix, $h_k^C = (h_k^1 ... h_k^m)^T$ is an $lm \times n$ -dimensional matrix.

The optimal estimate of vector $\hat{x}_k$ and the covariance matrix P_k can be calculated using the well-known recursive equations of the Kalman filter [15,16].

III. OVERVIEW OF RECURSIVE FEDERATED FILTERING ALGORITHMS

The above-mentioned feature of the problem associated with the presence, at each moment of time k, of a set of m measurements generated by local sensors, creates the prerequisites for designing algorithms based on the application of federated filters. As noted in the introduction, the main idea of the FFA is to obtain local estimates $\hat{x}_k^j$, calculated with only the use of measurements y_k^j, $j = 1...m$, and then, to form estimate $\hat{x}_k^G$, called global, by weighing the local estimates in the master filter. Let us explain the FFA in more detail.

In order to obtain local estimates $\hat{x}_k^j$ in the *j*-th LF, instead of Equation (1) for x_k, we introduce a local shaping filter for the sequence

$$x_k^j = \Phi_k^j x_k^j + \Gamma_k^j w_k^j, x_0^j \in N\{x^j; \bar{x}^j, P_0^j\}, w_k^j \in N\{w_k^j; 0, Q^j\}, \quad (4)$$

where Φ_k^j, Γ_k^j are the known $n \times p$ -dimensional matrices; w_k^j are the Gaussian zero-mean *p*-dimensional vectors of system noise with the known covariance matrices Q^j. It is assumed that v_k^j, w_k^j are white-noise sequences independent of each other and of x_0^j. Further, in each LF, local estimates and their covariance matrices $\hat{x}_k^j, P_k^j$ are designed. The global estimate $\hat{x}_k^G$ and its covariance matrix P_k^G in the master filter are calculated at time *k* as

$$P_k^G = (\sum_{j=1}^m \left(P_k^j\right)^{-1})^{-1}, \quad (5)$$

$$\hat{x}_k^G = P_k^G \sum_{j=1}^m \left(P_k^j\right)^{-1} \hat{x}_k^j. \quad (6)$$

It is known [1,2,6,7] that, in the general case, the estimate and the covariance matrix calculated in this manner will be different from the optimal estimate and covariance matrix. In [1,2], the authors formulate the following conditions for LF tuning

$$\sum_{j=1}^m \left(P_{k/k-1}^j\right)^{-1} = \left(P_{k/k-1}\right)^{-1}, \quad (7)$$

$$\sum_{j=1}^m \left(P_{k/k-1}^j\right)^{-1} \hat{x}_{k/k-1}^j = \left(P_{k/k-1}\right)^{-1} \hat{x}_{k/k-1}, \quad (8)$$

where $P_{k/k-1}^j$, $P_{k/k-1}$ are the covariance matrices of prediction errors in the *j*-th LF and the centralized filter (CF); $\hat{x}_{k/k-1}^j, \hat{x}_{k/k-1}$ are predictions in the *j*-th LF and CF. When the conditions (7)-(8) are met, the global estimate (6) and covariance matrix (5) will coincide with the optimal estimate and the covariance matrix generated in the CF.

The conditions (7), (8) are difficult to use directly when FFAs are implemented in practice. As known, in FFAs without reset of LFs, parameters $P_{k/k-1}^j$ and $\hat{x}_{k/k-1}^j$ are not recalculated after the global estimate is obtained in the master filter, as is done in FFAs with reset of LFs. Therefore, the parameters that affect the calculation of $P^j_{k/k-1}$ and $\hat{x}^j_{k/k-1}$ and can be used for FFA tuning are matrices Φ^j_k, Γ^j_k, P^j_0 and Q^j. Note that usually, when considering recursive FFAs, it is assumed by default that for all $j = 1...m$: $\Phi^j_k = \Phi_k$, $\Gamma^j_k = \Gamma_k$ and $\bar{x}^j = \bar{x}$, thus, FFAs tuning is reduced to choosing matrices P^j_0, and Q^j. It should also be emphasized that FFAs tuning results in the fact that the estimates obtained in the LFs will be different from the optimal local estimates.

Among all possible LF parameter setups, the so-called consistent setting is most preferential [2,9]; in this case, the estimate $\hat{x}^G_k$ and the covariance matrix P^G_k coincide with the optimal estimate $\hat{x}_k$ and the corresponding covariance matrix P_k obtained with the use of the centralized processing scheme.

It is known [1,2] that it is generally impossible to ensure a consistent tuning by choosing matrices P^j_0 and Q^j. In this regard, it is recommended to reduce the FFA errors, as compared to the optimal CF, by the following information-sharing conditions:

$$\sum_{j=1}^{m}\left(P^j_0\right)^{-1} = P_0^{-1}; \quad \sum_{j=1}^{m}\left(Q^j\right)^{-1} = Q^{-1}. \qquad (9)$$

Although, these conditions do not ensure the coincidence of $\hat{x}^G_k$ with the optimal estimate, they allow obtaining a guaranteed accuracy of FFA estimation in the sense of inequality

$$P^G_k > D^G_k \geq P_k , \qquad (10)$$

where D^G_k is the real covariance matrix of the FFA estimate.

Conditions (9) can also be written in a more convenient form, using information-sharing factors β^j [3-8]:

$$\left(P^j_0\right) = \beta^j P_0 ; \left(Q^j\right) = \beta^j Q ; \qquad \sum_{j=1}^{m}\left(\beta^j\right)^{-1} = 1 . \qquad (11)$$

Note that conditions (9) (or (11)) allow a certain freedom in choosing the settings of the LFs.

As noted, in the general case, conditions (9) do not provide a consistent tuning of the LF [1,2,9]. And at last, for special cases, such as estimation of time-invariant vector x_k, it can be shown (Appendix 1) that conditions (9) are sufficient to obtain an optimal estimate by FFAs.

Also, in [2], for the problem considered in this paper, it is shown that consistent tuning can be ensured provided that, in addition to (9), the following condition is met:

$$Q^j_k = \Phi_k P^j_k P_k^{-1} \Phi_k^{-1} Q . \qquad (12)$$

In this case, matrices Q^j_k will not only be variable, but also asymmetric; for this reason, it is difficult to fulfill the additional condition (12) in practice.

We should also note the case of the random walk estimation using scalar white-noise measurements. In this case, the consistent tuning of the LFs is achieved by fulfilling, in addition to (9), the following conditions for the information-sharing factors β^j [9]:

$$\left(\beta^1\right)\left(R^1\right)^{-1} = \left(\beta^2\right)\left(R^2\right)^{-1} ... = \left(\beta^m\right)\left(R^m\right)^{-1} . \qquad (13)$$

To sum up, it can be noted that for the problem under consideration, to obtain an optimal estimate in the recursive FFA, it is not sufficient to comply with the tuning requirements for (9). Additional requirements for (12) or (13) are difficult to implement in practice. Besides, to satisfy condition (13) requires taking into account the variances of measurement noise, which significantly narrows the number of options for possible settings of LFs.

nonrecursive federated filtering algorithms

Let us now specify the statement of the estimation problem under consideration as applied to a nonrecursive scheme.

First, consider the CF. Following [14], we form a composite *lmk*-dimensional vector of measurements $Y = ((y^C_1)^T .....(y^C_k)^T)$ accumulated by time *k* and a composite *n(k+1)*-dimensional state vector $X = (x^T_0,...,x^T_k)^T$ to be estimated, including all components x_k for $i = \overline{0.k}$. In this case, the problem of CF estimation can be formulated as follows: estimate the composite vector X by the measurements accumulated by time *k:*

$$Y = HX + V , \; V \in N\left\{V; 0, \mathbf{R}^{CP}\right\} , \qquad (14)$$

in which H - block matrix of $mlk \times n(k+1)$; $\mathbf{R}^{CP} = diag\left\{0_{lm\times lm}, R^C,..R^C\right\}$ is a matrix of $lmk \times lmk$ dimension; $V = (0,(v^C_1)^T,...,(v^C_\kappa)^T)^T$ is an *lmk*-dimensional vector. Here and below $0_{c\times p}$ is a zero $c \times p$-dimensional matrix.

In this case, the optimal estimate of the composite vector $\hat{X}(Y)$ and its covariance matrix $\mathbf{P}$ can be calculated as

$$\hat{X}(Y) = \bar{X} + \mathbf{P}H^T(\mathrm{R}^{CP})^{-1}\left(Y - H\bar{X}\right), \qquad (15)$$

$$\mathbf{P} = \left(\mathbf{P}_X^{-1} + H^T(\mathbf{R}^{CP})^{-1}H\right)^{-1}, \qquad (16)$$

where $\mathbf{P}_X, \bar{X}$ are a priori covariance matrix of estimation errors and mathematical expectation corresponding to the composite vector X [14]. Estimate $\hat{x}_k$ and covariance matrix P_k, which we need, are easy to obtain by extracting the corresponding component from vector $\hat{X}$ and the block from the covariance matrix $\mathbf{P}$.

For what follows, it is important that the problem of an *n*-dimensional sequence estimation at the *k*-th time moment is reduced to the problem of estimating a time-invariant *n(k+1)*-dimensional vector.

Now, let us specify the estimation algorithm as applied to a nonrecursive scheme using FFA. In this case, in each LF, we need to estimate the composite vector $X^j = ((x^j_0)^T,...,(x^j_k)^T)^T$ of *n(k+1)* dimension, including the components of x^j_i for all time moments $i = \overline{0.k}$, using the *lk*-dimensional vector measurement $Y^j = ((y^j_1)^T,...,(y^j_k)^T)^T$:

$$Y^j = H^j X^j + V^j,\ V^j \in N\{V^j; 0, R^{jp}\},\ j \in \overline{1,m}, \quad (17)$$

where H^j is the matrix of $lk \times n(k+1)$ dimension; $V^j = ((v_1^j)^T, ..., (v_k^j)^T)^T$ is a composite lk-dimensional vector of measurement errors with a diagonal covariance matrix $R^{jp} = diag(R_1^j .... R_k^j)$ of $lk \times lk$ dimension.

Optimal estimates in the LFs of vectors X^j by measurement Y^j can be calculated using the formulas similar to (15)-(16):

$$\hat{X}^j(Y^j) = \bar{X}^j + \mathbf{P}^j (H^j)^T (R^{jp})^{-1} (Y^j - H^j \bar{X}^j), \quad (18)$$

$$\mathbf{P}^j = \left(\mathbf{P}_{X^j}^{-1} + (H^j)^T (R^{jp})^{-1} H^j\right)^{-1}, \quad (19)$$

where $\mathbf{P}_{X^j}, \bar{X}^j$ are the covariance matrices and mathematical expectations corresponding to the composite vectors X^j [14]. The global estimate and covariance matrix of the composite vector $X^G = ((x_0^G)^T, ..., (x_k^G)^T)^T$ are similar to (5),(6):

$$\mathbf{P^G} = (\sum_{j=1}^{m} (\mathbf{P}^j)^{-1})^{-1}, \quad (27)\ \hat{X}^G = \mathbf{P}^G \sum_{j=1}^{m} (\mathbf{P}^j)^{-1} \hat{X}^j. \quad (20)$$

After considering the problem of nonrecursive filtering as a problem of a time-invariant vector estimation for k=1 (Appendix 1), it is easy to see that to obtain an optimal estimate of the FFA, it is sufficient to satisfy the conditions:

$$\sum_{j=1}^{m} \mathbf{P}_{X^j}^{-1} = \mathbf{P}_X^{-1}, \qquad \bar{X}^j = \bar{X}. \quad (21)$$

In Appendix 2, we show that the fulfillment of (21) is ensured by satisfying the conditions:

$$\sum_j^m (P_0^j)^{-1} = (P_0)^{-1}; \sum_j^m (Q^j)^{-1} = Q^{-1}. \quad (22)$$

Let us emphasize the coincidence of conditions (22) for the nonrecursive FFA and conditions (9) for the recursive one. In so doing, note that (22) only provide consistent tuning for a non-recursive filter. Such a difference of the recursive scheme can be explained by the need to calculate the prediction and its covariance matrix, the calculation of which, as shown in [7], is performed taking into account not only matrices P_0^j and Q^j but also the matrix R^j. In the case of the nonrecursive scheme, no additional calculations are needed.

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IV. An Example

As an illustration, consider the estimation problem of a random walk using measurements with white-noise errors:

$$x_k = x_{k-1} + w_k, x_0 \in N\{x_0; 0, \sigma^2\}, w_k \in N\{w_k; 0, q\}, \quad (23)$$

using measurements

$$y_k^j = x_k^j + v_k^j; \quad v_k^j \in N\{v_k^j; 0, r^j\}; \ j = 1,2. \quad (24)$$

Here, x, y^j are scalar values, and σ^2, q, r^j are known variances.

For nonrecursive processing schemes, the corresponding vectors and matrices are written as follows:

$X = (x_0, ..., x_k)^T$, $Y = (y_1^1, y_1^2, ..., y_k^1, y_k^2)^T$, $V = (v_1^1, v_2^1, ..., v_k^1, v_k^2)^T$,

$H = (0_{2k\times 1} \ \ \tilde{h}), \tilde{h} = diag(h^C .... h^C), h^C = (1 \ \ 1)^T$,

$R^{CP} = diag(0, 0, r^1, r^2 .... r^1, r^2)$.

Respectively, for the j-th LF, we have

$X^j = (x_0^j, ..., x_k^j)^T$, $\bar{X}^j = 0$, $Y^j = (y_1^j, ..., y_k^j)^T$, $V^j = (0, v_1^j, ..., v_k^j)^T$,

$H^j = (0_{k\times 1} \ \ \tilde{h}^j), \tilde{h} = diag(h^j .... h^j)$, $h^j = 1, R^{jP} = diag(0 \ \ r^j ... r^j)$

Hence, the condition of consistent tuning (21) can be written as

$$\begin{vmatrix} \sum_{j=1}^{2}\left((\sigma^j)^{-2} + (q^j)^{-1}\right) & \cdots & 0 \\ -\sum_{j=1}^{2}(q^j)^{-1} & \cdots & \\ \vdots & \cdots & -\sum_{j=1}^{2}(q^j)^{-1} \\ 0 & \cdots & \sum_{j=1}^{2}(q^j)^{-1} \end{vmatrix} = \begin{vmatrix} (\sigma)^{-2} + (q)^{-1} & \cdots & 0 \\ -(q)^{-1} & \cdots & \\ \vdots & \cdots & -(q)^{-1} \\ 0 & \cdots & (q)^{-1} \end{vmatrix}. \quad (25)$$

It is clear that equality (23) is ensured when

$$(\sigma^1)^{-2} + (\sigma^1)^{-2} = (\sigma)^2; (q^1)^{-1} + (q^2)^{-1} = (q)^{-1} \quad (26)$$

We have already emphasized the coincidence of the obtained conditions for the consistent tuning with the conditions for the recursive FFAs scheme (9). But at the same time, we note that if only conditions (26) are used, the recursive scheme in this problem of the random walk estimation does not provide an optimal solution, as distinct from the nonrecursive scheme, for which these conditions are sufficient.

To illustrate the comparison of the recursive and nonrecursive FFAs in the problem of estimating the random walk (23) by measurements (24), we carry out the simulation under the following conditions: σ=10, q=3, r^1=1, and r^2=10.

The simulation results are shown in the f Fig. 1.

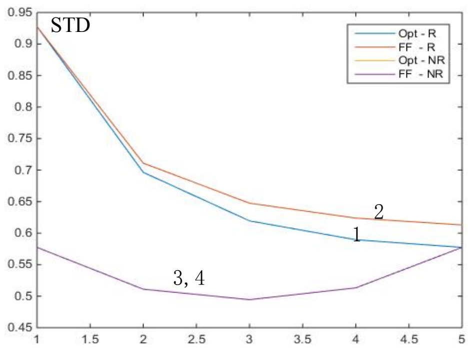

Fig. 1. The RMS errors of the recursive CF (Opt-R,1) and federated filter (FF-R,2) and the nonrecursive CF (Opt-NR,3) and federated filter (FF-NR,4) In FF-R $\beta^1 = \beta^2 = 2$. Note that Opt-NR and FF-NR coincide, but Opt-R and FF-R are different.

Figure 1 show the graphs of the standard deviation (STD) of the estimation error for the recursive scheme: CF (Opt-R) and federated filter (FF-R) and the coinciding graphs for the nonrecursive scheme: CF (Opt-NR) and (FF-NR). Note that, if the information-sharing factors are equal, (Fig. 1) the standard deviation of the recursive federated filter differs from the standard deviation of the recursive CF.

Thus, it is clear that the simulation results have confirmed the above conclusions about the possibility of obtaining an optimal estimate using FFAs for a nonrecursive scheme, whereas for a recursive scheme, such an opportunity occurs only if the additional conditions for the FF tuning (13) are taken into account.

The differences between the nonrecursive and recursive scheme, which allow obtaining an optimal estimate using FFAs, lie, first of all, in the fact that the result of the smoothing problem solution on the interval i=0..k in LFs is used to calculate a global estimate. On the contrary, in the recursive scheme, only the result of the filtering problem solution in LF at time k=1 is used.

V. CONCLUSION

The federated filtering methods have been considered as applied to the traditional recursive and nonrecursive schemes for measurement processing.

For the recursive scheme, a brief overview of the federated filtering algorithms is given and the conditions for the so-called consistent tuning of local filters are formulated. For special cases, it is shown that there are conditions for consistent tuning that make it possible to obtain an optimal solution using the recursive federated filtering algorithms. It is noted that it is generally impossible to calculate an optimal estimate by fusion of the estimates obtained in the recursive local filters.

The estimation problem solved with the use of federated filtering algorithms is formulated as applied to the nonrecursive scheme; the conditions for the consistent tuning of local filters are given. It has been shown that when the conditions of consistent tuning are met, the estimate obtained in the nonrecursive FF will coincide, in contrast to the recursive one, with the optimal estimate of the centralized Kalman filter for any type of the estimated process. It is noted that the main differences between the nonrecursive scheme and recursive one, which allow obtaining an optimal estimate, are due to result of the smoothing problem solution in local filters which is used to calculate the global estimate.

The results obtained are illustrated using a methodical example of the random walk estimation.

Appendix 1

Consider a special case of the problem of estimating the time-invariant vector

$$x_k = x_{k-1}, \quad x_0 \in N\{x_0; \bar{x}, P_0\}, \quad \text{(A1.1)}$$

using measurements

$$y_k^j = h^j x_k + v_k^j; \quad v_k^j \in N\{v^j; 0, R^j\}, j = 1..m \quad \text{(A1.2)}$$

Let us show that equality

$$\sum_{j=1}^{m} \left(P_0^j\right)^{-1} = P_0^{-1} \quad \text{(A1.3)}$$

ensures a consistent tuning in which estimate $\hat{x}_k^G$ and covariance matrix P_k^G of the FFA coincide with the estimate $\hat{x}_k$ and covariance matrix P_k calculated in the CF.

For k=1, with the assumptions made, it is easy to obtain an equation for the optimal estimate, calculated in the CF.

$$\hat{x}_1 = \bar{x} + P_1\left(\sum_{j=1}^{m}\left(h_1^j\right)^T (R^j)^{-1} y_1^j - \sum_{j=1}^{m}\left(h_1^j\right)^T (R^j)^{-1} h_1^j \bar{x}\right), \quad \text{(A1.4)}$$

and the corresponding covariance matrix

$$P_1 = \left(\left(P_0\right)^{-1} + \sum_{j=1}^{m}\left(h_1^j\right)^T (R^j)^{-1}\left(h_1^j\right)^T\right)^{-1}. \quad \text{(A1.5)}$$

Let us introduce the local estimates for model (A1.1) in the form:

$$\hat{x}_1^j = \bar{x}^j + P_1^j \left(h_1^j\right)^T (R^j)^{-1} (y_k^j - h_1^j \bar{x}^j), \quad \text{(A1.6)}$$

where $P_1^j = \left(\left(P_0^j\right)^{-1} + \left(h_1^j\right)^T \left(R^j\right)^{-1} h_1^j\right)^{-1}$.

The estimate and the covariance matrix in the FF can be calculated as

$$\hat{x}_1^G = P_1^G \sum_{j=1}^{m} ((P_1^j)^{-1} \bar{x}^j) + P_1^G \left(\sum_{j=1}^{m} (h_1^j)^T (R^j)^{-1} y_1^j - \sum_{j=1}^{m} (h_1^j)^T (R^j)^{-1} h_1^j \bar{x}^j\right) \quad \text{(A1.7)}$$

$$P_1^G = (\sum_{j=1}^{m} (P_1^j)^{-1})^{-1} = \left(\sum_{j=1}^{m} \left(P_0^j\right)^{-1} + \sum_{j=1}^{m} (h_1^j)^T (R^j)^{-1} h_1^j\right)^{-1}. \quad \text{(A1.8)}$$

It is easy to see that the FF global covariance matrix (A1.8) coincides with the covariance matrix of the optimal estimate (A1.5) if (A1.3) is satisfied. In this case, it is clear that the formula for the global estimate (A1.7) will also coincide with the formula for the optimal estimate of the CF (A1.4) when $\bar{x}^j = \bar{x}$.

Using method of the mathematical induction, it is easy to show that the same statements will also be true for any k>1.

Thus, it has been shown that equality (A1.3) ensures a consistent tuning, which provides the coincidence of estimates and the corresponding covariance matrices in the FFA with the optimal estimates and their covariance matrices. It should be emphasized that equality (A1.3) also ensures the fulfillment of condition (7):

$$\sum_{j=1}^{m} \left(P_{k/k-1}^j\right)^{-1} = \left(P_{k/k-1}\right)^{-1}, \quad \text{(A1.9)}$$

Thus, condition (9), which can be written as

$$\sum_{j=1}^{m} \left(P_k^j\right)^{-1} = \left(P_k\right)^{-1} \quad \text{(A1.10)}$$

is met, as was shown above, the equality $\left(P_k^G\right)^{-1} = \sum_{j=1}^{m} \left(P_k^j\right)^{-1} = \left(P_k\right)^{-1}$ is valid.

We will also show that condition (8) is satisfied. It can be written as:

$$\sum_{j=1}^{m}\left(P_k^j\right)^{-1}\hat{x}_k^j = \left(P_k\right)^{-1}\hat{x}_k . \quad (A1.11)$$

If we take into account $\hat{x}_k^G = P_k^G \sum_{j=1}^{m}\left(P_k^j\right)^{-1}\hat{x}_k^j$ and $\hat{x}_k^G = \hat{x}_k$, we can write:

$$\sum_{j=1}^{m}\left(P_k^j\right)^{-1}\hat{x}_k^j = \left(P_k^G\right)^{-1} P_k^G \sum_{j=1}^{m}\left(P_k^j\right)^{-1}\hat{x}_k^j = \left(P_k\right)^{-1}\hat{x}_k . \quad (A1.12)$$

Appendix 2

Let us find the conditions for fulfillment of the equality $\sum_{j=1}^{m}\mathbf{P}_{X^j}^{-1} = \mathbf{P}_X^{-1}$. Matrix $\boldsymbol{P_X}$, as was shown in [14], can be determined as $\boldsymbol{P_X} = \boldsymbol{T P^r T^T}$, where

$$\boldsymbol{T} = \begin{vmatrix} E & 0 & \cdots & 0 \\ \Phi_1 & \Gamma_1 & \ddots & \vdots \\ \ddots & \ddots & \ddots & 0 \\ \Phi_k\Phi_{k-1}.....\Phi_1 & \Phi_1\Phi_1.....\Phi_2\Gamma & \Phi_k & \Gamma_k \end{vmatrix}; \quad \boldsymbol{P^r} = \begin{vmatrix} P_0 & 0 & \cdots & 0 \\ 0 & \Gamma_1 Q_1 \Gamma_1^T & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma_k Q_k \Gamma_k^T \end{vmatrix}. \quad (A2.1)$$

Then, using the inversion rule for triangular matrices, matrix $\left(\boldsymbol{P_X}\right)^{-1}$ can be represented as

$$\left(\boldsymbol{P_X}\right)^{-1} = \left(\boldsymbol{T P^r T^T}\right)^{-1} = \left(\boldsymbol{T^T}\right)^{-1}\left(\boldsymbol{P^r}\right)^{-1}\left(\boldsymbol{T}\right)^{-1} = \boldsymbol{F}^T \boldsymbol{D F}, \quad (A2.2)$$

where $\boldsymbol{F} = \boldsymbol{T}^{-1} = \begin{vmatrix} E & 0 & \ldots & 0 \\ -\Phi_1 & \left(\Gamma_1\right)^{-1} & \ddots & \vdots \\ \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\Phi_k & \left(\Gamma_k\right)^{-1} \end{vmatrix}$,

$$\boldsymbol{D} = \left(\boldsymbol{P^r}\right)^{-1} = \begin{vmatrix} P_0^{-1} & 0 & \ldots & 0 \\ 0 & \left(\Gamma_1 Q_1 \Gamma_1^T\right)^{-1} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \left(\Gamma_k Q_k \Gamma_k^T\right)^{-1} \end{vmatrix}. \quad (A2.3)$$

If we assume $\Gamma = E$, we can write:

$$\left(\boldsymbol{P_X}\right)^{-1} = \begin{vmatrix} P_0^{-1} + \Phi_1^T\left(Q_1\right)^{-1}\Phi_1 & -\Phi_1^T\left(Q_1\right)^{-1} & \cdots & 0 \\ -\left(Q_1\right)^{-1}\Phi_1 & \left(Q_1\right)^{-1} & \ddots & \vdots \\ \ddots & \ddots & \ddots & -\Phi_k^T\left(Q_k\right)^{-1} \\ 0 & 0 & -\left(Q_k\right)^{-1}\Phi_k & \left(Q_k\right)^{-1} \end{vmatrix}. \quad (A2.4)$$

Matrix $\left(\boldsymbol{P_{X^j}}\right)^{-1}$ can be written in a similar way, so that condition $\sum_{j=1}^{m}\mathbf{P}_{X^j}^{-1} = \mathbf{P}_X^{-1}$ is reduced to equality:

$$\begin{vmatrix} \sum_j^m\left(P_0^j\right)^{-1} + \sum_j^m\left(\Phi_1^j\right)^T\left(Q_1^j\right)^{-1}\Phi_1^j & -\sum_j^m\left(\Phi_1^j\right)^T\left(Q_1^j\right)^{-1} & & 0 \\ -\sum_j^m\left(Q_1^j\right)^{-1}\Phi_1^j & \sum\left(Q_1^j\right)^{-1} & \ldots & -\sum_j^m\left(\Phi_k^j\right)^T\left(Q_k^j\right)^{-1} \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & & \sum_j^m\left(Q_k^j\right)^{-1} \end{vmatrix}$$

$$= \begin{vmatrix} P_0^{-1} + \Phi_1^T\left(Q_1\right)^{-1}\Phi_1 & -\Phi_1^T\left(Q_1\right)^{-1} & \cdots & 0 \\ -\left(Q_1\right)^{-1}\Phi_1 & \left(Q_1\right)^{-1} & \ddots & \vdots \\ \ddots & \ddots & \ddots & -\Phi_k^T\left(Q_k\right)^{-1} \\ 0 & 0 & -\left(Q_k\right)^{-1}\Phi_k & \left(Q_k\right)^{-1} \end{vmatrix}, \quad (A2.5)$$

which allows the following conditions for the consistent tuning of the local filters to be written as

$$\left.\begin{aligned} &\sum_j^m\left(P_0^j\right)^{-1} = \left(P_0\right)^{-1}; \sum_j^m\left(Q_k^j\right)^{-1} = Q_k^{-1}; \\ &\sum_j^m\left(Q_{k-1}^j\right)^{-1}\Phi_k^j = \left(Q_{k-1}\right)^{-1}\Phi_k; \sum_j^m\left(\Phi_k^j\right)^T\left(Q_{k-1}^j\right)^{-1} = \Phi_k^T\left(Q_{k-1}\right)^{-1}; \\ &\sum_j^m\left(\Phi_1^j\right)^T\left(P_0^j\right)^{-1}\Phi_1^j + \left(Q_1^j\right)^{-1} = \Phi_1^T P_0^{-1}\Phi_1 + Q_1^{-1}. \end{aligned}\right\} \quad (A.2.6)$$

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