

# Synchronization Analysis of Circadian Rhythms Using Kim-Forger Dynamics

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**Abstract**—Circadian rhythms, which follow a roughly 24-hour cycle, are crucial for regulating sleep-wake cycles, hormone secretion, and other physiological processes. Disruptions in these rhythms are linked to various mental health disorders, including depression, anxiety, and panic disorders. Understanding and synchronizing these rhythms requires analyzing coupled non-linear systems. The coupled Kim-Forger oscillator dynamics, a non-linear system represented by ordinary differential equations, better describe circadian rhythms dynamics. Synchronizing these oscillators involves adding coupling terms, but determining the precise amount of coupling required can be challenging. Contraction theory offers a framework for ensuring robust synchronization by analyzing the stability and convergence of dynamical systems. This paper explores the application of contraction theory to the Kim-Forger oscillator model, aiming to identify sufficient conditions for synchronization (at coupling point=0.6). Numerical simulations validate the results, highlighting the potential of this approach to normalize disrupted circadian rhythms and alleviate related mental health symptoms.

Circadian Rhythms, Kim-Forger Oscillator Model, Contraction Theory, Synchronization, Mental Health Disorders

## I. INTRODUCTION

Circadian rhythms are endogenous, entrainable oscillations that regulate various physiological processes and maintain homeostasis in living organisms. These rhythms follow an approximately 24-hour cycle, aligning the body's internal processes—such as sleep-wake cycles, hormone release, and body temperature regulation—with environmental cues, particularly light and darkness. At the core of circadian rhythms lies the circadian clock, a complex regulatory system comprising transcriptional-translational feedback loops (TTFLs) involving clock genes and proteins [1]. This system ensures that biological processes are synchronized with external environmental cycles, thereby promoting optimal physiological function.

Circadian rhythm disruptions can have a significant impact on health, especially on sleep. Sleep is among most prominent expressions of circadian rhythms, and its regulation is tightly controlled by the interaction between the sleep homeostat and the circadian system [2]. The disruption of circadian rhythms—due to factors such as irregular light exposure, shift work, jet lag, or lifestyle changes—can lead to a misalignment

between the body's internal clock and external environmental cues, resulting in poor sleep quality, insomnia, and other sleep disorders. Circadian misalignment is increasingly acknowledged as a contributing cause to psychiatric diseases, including depression, anxiety, and bipolar disorder [3].

At the molecular level, circadian rhythms are regulated by feedback loops involving clock genes and proteins. These loops generate self-sustained oscillations that control the timing of various biological processes. Traditional models, such as those based on Hill-type functions, describe circadian transcriptional repression using a mathematical framework that accounts for gene expression dynamics [4]. However, a novel class of models has emerged, focusing on protein sequestration as a mechanism for circadian regulation [5]. These models provide an alternative perspective, deviating significantly from Hill-type repression models in terms of network architectures, oscillator periods, and rhythm-generating circumstances.

Several key factors can disrupt circadian rhythms and negatively impact sleep. Light exposure, especially blue light, is a major environmental factor, as it delays melatonin release and shifts the circadian clock, leading to delayed sleep onset and reduced sleep duration when exposed to artificial lighting and electronic devices in the evening [6]. Referring to fig.1 shows that irregular schedules, such as shift work and inconsistent sleep patterns, contribute to circadian misalignment, causing sleep disturbances and potential long-term health issues [7]. Social jetlag, which occurs when there is a misalignment between an individual's biological clock and social obligations—such as staying up late and sleeping in on weekends—can lead to sleep deprivation and circadian disruption when attempting to return to a weekday schedule [8]. Additionally, psychological stress and mood disorders, including anxiety and depression, can destabilize circadian rhythms, further complicating recovery and treatment [8].

Synchronizing circadian rhythms with environmental cues, such as the natural light-dark cycle, is essential for maintaining physiological homeostasis and promoting optimal health. Proper alignment of the circadian clock ensures the timely release of hormones such as melatonin, which regulates sleep,

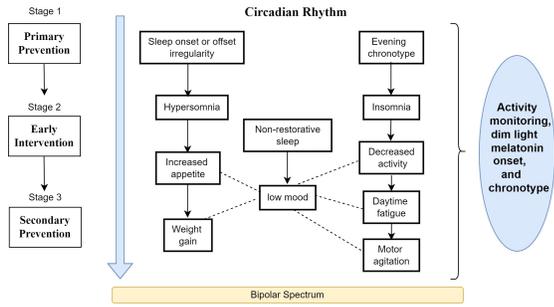


Fig. 1. Activity monitoring flow chart based on circadian rhythm

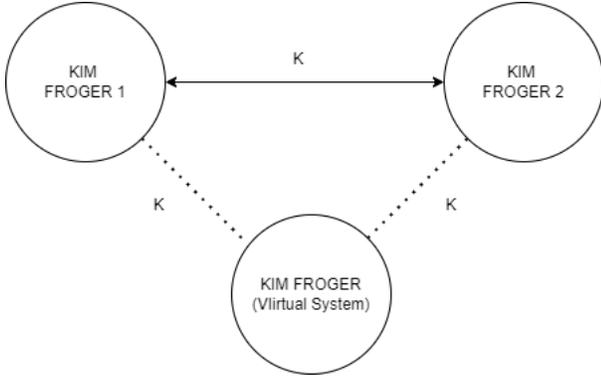


Fig. 2. Synchronization of coupled Kim-Froger oscillators

and cortisol, which governs stress responses. When circadian rhythms are synchronized, individuals experience better sleep quality, enhanced cognitive function, and improved overall well-being [9].

Circadian rhythm disruptions have been linked to a number of health concerns, such as mental health issues, cardiovascular illness, and metabolic diseases. The importance of restoring circadian synchrony has become a focus of research aimed at developing therapeutic interventions. For example, contraction theory has been applied to mathematical models like the Kim-Froger oscillator model to ensure the stability and synchronization of circadian oscillators [5]. These models offer valuable insights into the mechanisms underlying circadian regulation and highlight potential therapeutic strategies for addressing circadian disruptions.

## II. CONTRACTION THEORY FOR DYNAMICAL SYSTEMS

The contraction has been proven to be a beneficial feature for analyzing network convergence and studying problems like the development of synchronization or consensus [21 – 24]. All trajectories of a contracting system may be demonstrated to converge asymptotically towards each other exponentially. As demonstrated in [21], this characteristic may be used to provide criteria for synchronizing a network of dynamical systems of interest. Non-Euclidean matrix measurements have been shown to be beneficial in developing an algorithmic strategy to prove contraction and efficient convergence in

biological networks. In addition, a hierarchical technique for studying convergence through a contraction was developed. The authors in [24] suggests the utilization of a distributed discontinuous coupling technique to attain convergence and synchronization in networks of non-homogeneous nonlinear dynamical systems. Authors have addressed several techniques for studying network convergence and synchronization using contraction analysis in [25].

The contracting dynamical systems inherently exhibit the following properties:

- (i) The exponential convergence is guaranteed.
- (ii) The contracting dynamical systems are highly robust.
- (iii) They have finite input-to-state gain.
- (iv) Contraction margin to unmodeled dynamics exists
- (v) Input to state stability in delayed dynamics is ensured.

Therefore, Our current study focuses on how much coupling increase is needed to synchronize coupled Kim Froger systems. A study based on contraction theory has been used to calculate the necessary coupling gain. The goal is to give sharp requirements for synchronization and exponential convergence in coupled Kim-Froger oscillators. Our main contributions are listed below.

- (i) Using the contraction theory framework, we calculate the sufficient coupling gain required for the synchronization of Kim-Froger oscillators.
- (ii) Through Numerical simulations, we confirm the analytical findings

The following section presents the Problem Statement and Synchronization Analysis; Section IV reports the Numerical simulations and the paper is summarized in Section V.

## III. PROBLEM STATEMENT AND SYNCHRONIZATION ANALYSIS

The problem statement is *To derive sufficient coupling gain for synchronization of coupled Kim-Froger oscillators using Contraction theory framework*

### SYNCHRONIZATION ANALYSIS

#### A. How to synchronize biological oscillators ?[25]

The process of synchronization of biological clocks has been well illustrated in the work [62].

- (i) Consider two coupled non-linear systems

$$\dot{q} = g(q) + (h(p) - h(q)) \quad (1)$$

$$\dot{p} = g(p) + (h(q) - h(p)) \quad (2)$$

- (ii) Choose a suitable virtual system,

$$\dot{r} = g(r) + (h(q) - h(p)) + (h(p) - h(r)) \quad (3)$$

This virtual system relies on the state variables of the network nodes, specifically q and p, as well as the virtual state variable r. Substituting q and p for r in equation (3) results in equations (1) and (2). Therefore, the trajectories q and p represent specific solutions of  $\dot{r}$ . The contracting property of the virtual system concerning the virtual state variable indicates that all solutions converge to one

another. Consequently, in a contracting virtual system, the specific r solutions q and p exhibit convergence towards one another. This ensures the synchronization of q and p. Thus, if we can show that system 3 is contracting, it is equivalent to showing synchronization of (1) and (2).

- (iii) In doing so, the Jacobian of (3) is required to be evaluated.
- (iv) The contractivity of the virtual system can be assured by ensuring that the symmetric part of the Jacobian is uniformly negative definite, which involves imposing the negativity of the eigenvalues of the system Jacobian.

### B. Synchronization Analysis of Coupled Kim-Forger Oscillators

The synchronization analysis presented here is based on [52]. The dynamics of 1<sup>st</sup> oscillator be given as,

$$\dot{x}_1 = K_1 f(z_1) - K_4 x_1 + K(x_2 - x_1) \quad (4)$$

$$\dot{y}_1 = K_2 x_1 - K_5 y_1 + K(y_2 - y_1) \quad (5)$$

$$\dot{z}_1 = K_3 y_1 - K_6 z_1 + K(z_2 - z_1) \quad (6)$$

Where,  $K_1 = K_2 = K_3 = 1, K_4 = 0.16, K_5 = 0.29, K_6 = 0.3, K_7 = 0.6, K_8 = 10^{-5}$  and  $f(z_1) = \frac{K_7 - K_8 + \sqrt{(K_7 - K_8 - z)^2 + 4K_7 K_8}}{2K_7}$ . Similarly, the dynamics of 2<sup>nd</sup> oscillator be given as,

$$\dot{x}_2 = K_1 f(z_2) - K_4 x_2 + K(x_1 - x_2) \quad (7)$$

$$\dot{y}_2 = K_2 x_2 - K_5 y_2 + K(y_1 - y_2) \quad (8)$$

$$\dot{z}_2 = K_3 y_2 - K_6 z_2 + K(z_1 - z_2) \quad (9)$$

The dynamics of a virtual system may be given as,

$$\dot{\hat{x}} = K_1 f(\hat{z}) - K_4 \hat{x} + K(x_1 - \hat{x}) + K(x_2 - \hat{x}) \quad (10)$$

$$\dot{\hat{y}} = K_2 \hat{x} - K_5 \hat{y} + K(y_1 - \hat{y}) + K(y_2 - \hat{y}) \quad (11)$$

$$\dot{\hat{z}} = K_3 \hat{y} - K_6 \hat{z} + K(z_1 - \hat{z}) + K(z_2 - \hat{z}) \quad (12)$$

Thus, the Jacobian Matrix of the virtual system is,

$$J(V) = \begin{bmatrix} -K_4 - 2K & 0 & K_1 \frac{\partial f(\hat{z})}{\partial z} \\ K_2 & -K_5 - 2K & 0 \\ 0 & K_3 & -K_6 - 2K \end{bmatrix}$$

The matrix measure  $\mu_1(J(V))$  is defined as,

$$\mu_1(J(V)) = \max_j [J_{jj} + \sum_{i \neq j} |J_{ij}|] \quad (13)$$

$$\text{or, } \mu_1(J(V)) = \text{Max}[-K_4 - 2K + |K_2|; -K_5 - 2K + |K_3|; -K_6 - 2K + |K_1 \frac{\partial f(\hat{z})}{\partial z}|] \quad (14)$$

Thus, for synchronization or contractivity  $\mu_1(J(V)) < 0$ . This is possible if

- (i)  $K > 0.5K_2$
- (ii)  $K > 0.5K_3$
- (iii)  $K > 0.5 \frac{\partial f(\hat{z})}{\partial z} = 0.5 \left[ \frac{-1}{2K_7} + \frac{1}{2K_7} \left[ \frac{2(k_7 - K_8 - z)}{2\sqrt{(K_7 - K_8 - z)^2 + 4K_7 K_8}} \right] \right]$

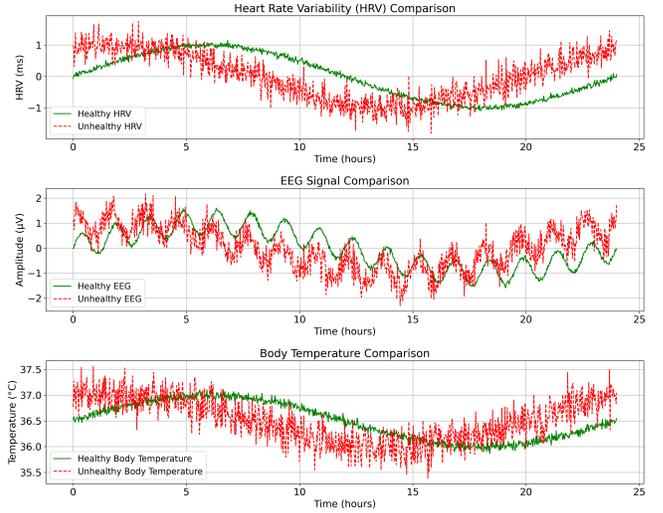


Fig. 3. Comparative Analysis of Circadian Rhythm Factors: HRV, EEG, and Body Temperature in Healthy vs. Unhealthy Subject

Now,  $\frac{\partial f(\hat{z})}{\partial z} = 0$ . Thus, satisfying  $K > 0.5$  yields synchronization. The numerical simulation confirming the above result is presented in Fig.5.

### IV. IMPLICATIONS OF DE-SYNCHRONISED CIRCADIAN RHYTHMS

Desynchronized circadian rhythms significantly impact sleep, leading to various sleep-related issues such as insomnia, delayed sleep phase disorder (DSPD), irregular sleep-wake rhythm, advanced sleep phase disorder (ASPD), and shift work sleep disorder (SWSD). These disruptions result from the misalignment between an individual's internal body clock and external environmental cues, causing difficulties in maintaining consistent and restorative sleep patterns. Chronic sleep deprivation, a common consequence of these disorders, impairs cognitive function, mood regulation, and overall health, highlighting the critical need for effective diagnostic and therapeutic interventions. Fig.3 compares the rhythmic patterns of heart rate variability (HRV), EEG, and body temperature between a healthy and unhealthy subject over a 24-hour cycle (here the signals are purely synthetically generated for understanding purpose). In the healthy subject, all three factors—HRV, EEG, and temperature—demonstrate clear, smooth oscillations in sync with the circadian rhythm, reflecting well-regulated biological cycles. These patterns suggest strong circadian alignment, characterized by stable fluctuations in physiological processes. In contrast, the unhealthy subject shows irregular, noisier data with phase shifts and erratic fluctuations, indicating circadian desynchronization. This disruption in rhythmic patterns is typically seen in conditions such as sleep disorders or chronic stress, where the body's natural cycles are misaligned, leading to poorer physiological regulation.

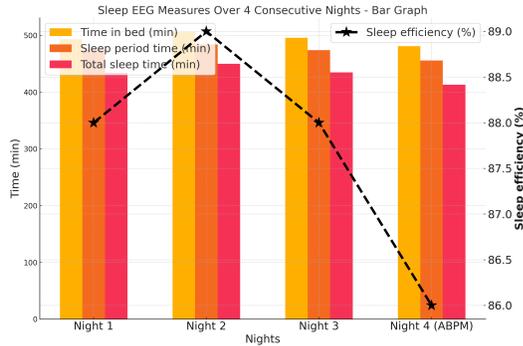


Fig. 4. Sleep Efficiency Variations Over Four Consecutive Nights with Ambulatory Monitoring for 31 healthy subjects referred from [63]

The Fig.4 illustrates the sleep patterns of 31 healthy young men over four consecutive nights, highlighting key parameters such as time in bed, sleep period time, total sleep time, and sleep efficiency. The data shows that while the time in bed, sleep period time, and total sleep time remain relatively stable over the first three nights, all three measures decrease by Night 4, which involved ambulatory blood pressure monitoring (ABPM). Sleep efficiency, represented by the black dotted line, remains around 88% – 89% for the first three nights, but drops significantly on Night 4 to 86%, indicating a potential disruption in sleep quality [62].

The changes in sleep efficiency and total sleep time on Night 4 suggest that external factors, such as the ABPM monitoring, may have interfered with the subjects' sleep. This demonstrates the sensitivity of sleep patterns to external conditions, underscoring the importance of maintaining a stable sleep environment for optimal rest. In the context of circadian rhythms, this data highlights how consistency in sleep duration and quality is crucial for overall well-being, as disruptions can affect physiological and cognitive function.

## V. NUMERICAL SIMULATIONS

To verify the theoretical findings, we carry out numerical simulations using the ode45 solver in MATLAB. The parametric values chosen for performing simulation were  $k_1=1; k_2=1; k_3=1; k_4=0.16; k_5=1; k_6=0.3; k_7=0.6; k_8=10^{-8}; K = 0.6$ ; It is important to mention that for the initial time span of 99 seconds, the coupling gain was off, while at  $time \geq 100$  seconds, the value of the coupling gain was chosen to be  $K=0.6$ . Fig.2, Fig.3, Fig.4 and Fig.5 show simulations depicting the synchronization in two coupled Kim-Froger Oscillators with coupling gains  $K=0.6, K=0.8, K=1, K=10$  respectively. It was observed that the coupled Kim-Froger oscillators were synchronized after a short transient. This validated and confirmed our theoretical findings.

## VI. CONCLUSION

In the present work, we aim to compute sufficient coupling gain for synchronizing coupled Kim-Froger dynamics. Here, we use the contraction theory framework to calculate the sufficient coupling gain. In this work, we use 1-norm

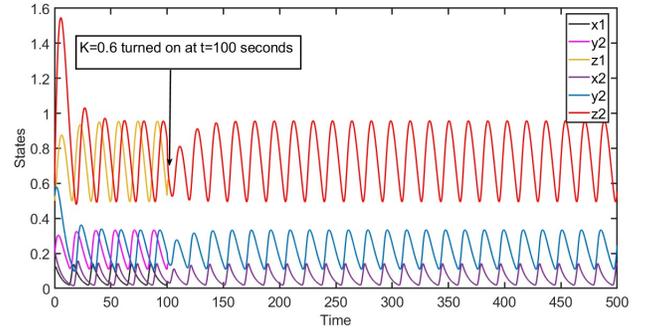


Fig. 5. Synchronization of coupled Kim-Froger oscillators

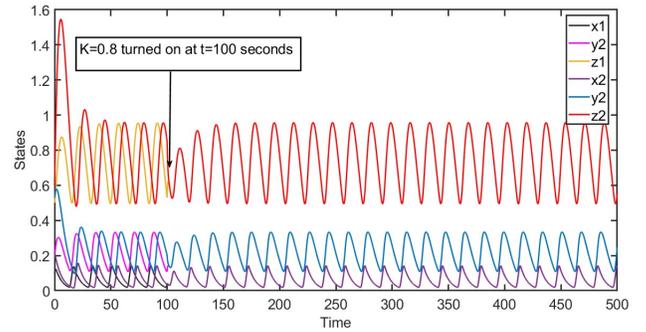


Fig. 6. Synchronization of coupled Kim-Froger oscillators

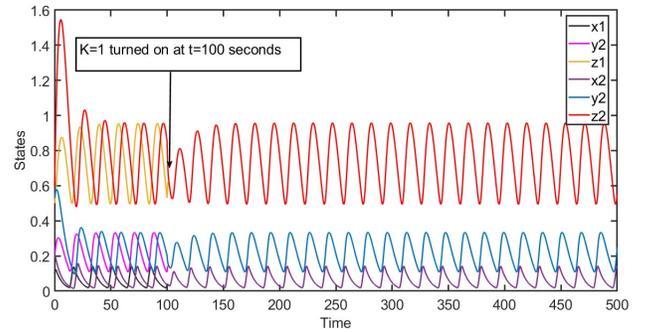


Fig. 7. Synchronization of coupled Kim-Froger oscillators

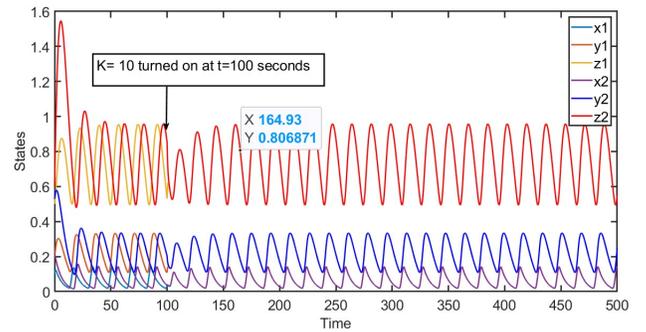


Fig. 8. Synchronization of coupled Kim-Froger oscillators

as the matrix measure. Using the state-of-the-art approach described in the literature, we demonstrate synchronization in a benchmark model of circadian rhythms named the Kim-Froger oscillator. We also provide a state-of-the-art literature survey on the contraction theory framework for stability analysis of dynamical systems. Contraction theory is a promising tool to study the stability of dynamical systems. Many of its key features were also addressed. Future research can be carried out to investigate the synchronization of coupled Kim-Froger oscillators that are connected in arbitrary connection topologies. Further, this work can be extended to other types of complex networks of benchmark oscillatory systems that play a vital role in nature.

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