Finite-Time Synchronization of Master-Slave Chaotic Systems with Constant Time Delays*

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***Abstract*— This paper presents a finite-time synchronization framework for a class of nonlinear systems with constant delays. The control strategy utilizes the backstepping technique to design a control input that drives the synchronization error between the master and slave systems to zero within finite time. This framework ensures fast convergence while accounting for the challenges posed by the time delays. The structure of the generalized nonlinear system enables the application of this approach to a wide range of chaotic systems. In particular, in this paper, the Genesio-Tesi chaotic system and Sprott Circuit system are employed to validate the proposed methodology through simulation. The resulting simulations clearly demonstrate the effectiveness of the proposed control strategy, confirming its capability to achieve finite-time synchronization.**

I. INTRODUCTION

Chaotic systems play a crucial role in a wide range of fields, including mathematics, life sciences, biomedical applications, cryptography, spread spectrum waveform design and secure communications. Due to their widespread presence and significance, it is essential to understand these systems for effective control and management. Consequently, the challenge of controlling chaos has attracted significant interest from the research community across multiple disciplines. A landmark development in this field was done in [1], laying the groundwork for synchronization of chaotic systems. Since then, extensive research has been carried out to synchronize various chaotic and hyperchaotic systems. Some notable strategies such as active control [2], sliding mode control [3], nonlinear feedback control [4], adaptive control strategies [5], observer-based approaches [6] have been explored to facilitate synchronization.

Among the various control techniques, the backstepping method [7], [8] is widely recognized for its recursive and systematic approach to design stabilizing and synchronizing controllers, particularly when dealing with strict-feedback system structures. For systems not inherently in this structure, a change of variables can transform their dynamics to enable backstepping method. This method relies on Lyapunov stability theory.

In practical engineering applications, many systems require stabilization or synchronization within a finite or predetermined time, making synchronization speed a critical factor. Finite-time control has gained prominence as an effective approach to achieve rapid convergence. Recent research has addressed finite-time stabilization for various classes of nonlinear systems, including higher-order [9], non-autonomous [10], time-delay [11], switched [12], and lower-triangular [13] systems. Further, finite-time stabilization using backstepping [14], dynamic high-gain control [15], output-feedback strategies [16], control vector Lyapunov functions [17], terminal sliding mode [18], etc. have been extensively explored for achieving finite-time stabilization in different classes of nonlinear systems.

However, despite their effectiveness, these approaches often face structural limitations and involve complex controller design processes. The primary contribution of this work lies in introducing a finite-time synchronization framework that applies to a broader category of nonlinear systems with constant delays. Unlike existing approaches that ignores time delays [19], [20] this work explicitly incorporates time-delay effects into both the master and slave system dynamics through delayed linear terms. We propose a synchronization framework that remains robust against such time delays. The work done in [21] proposes synchronization of uncertain chaotic systems under time-varying time delays. However, its proposed method is tailored to a specific nonlinear structure, thus the applicability of the approach remains limited to a narrow class of system. In contrast, the generalized formulation presented here encompasses several well-known chaotic systems, including the Genesio-Tesi system [22], Sprott Circuits [23], Duffing [24] and Van der Pol [25] oscillators, thereby extending the applicability of the proposed control strategy. A key novelty lies in designing a control input that guarantees finite-time convergence of the synchronization error to zero, despite the presence of constant delays, thus ensuring rapid and reliable synchronization within a finite-time.

This paper is structured as follows: Section II provides the basic concepts and lemmas about finite-time stability. Section III gives the problem formulation. The main results are described in detail in Section IV. Numerical Simulation validating the main results are given in Section V while Section VI concludes this paper.

II. PRELIMINARIES

Consider an open connected set $S \subseteq \mathbb{R}^n$ that includes the origin. Let us examine a nonlinear system described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1}$$

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where $\boldsymbol{x}$ represents the $n-$dimensional state vector. The function $\boldsymbol{f}:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a vector function with $\boldsymbol{f}(\mathbf{0})=\mathbf{0}$. We say that the origin of system (1) is Lyapunov stable if, for any $\boldsymbol{x}(t_0)\in S$, the solution $\boldsymbol{x}(t;\boldsymbol{x}_0)$ exists $\forall\ t\geqslant 0$, and for any $\epsilon>0$, there exists a $\delta>0$ such that if $||\boldsymbol{x}_0||\leqslant\delta$ for any $\boldsymbol{x}_0\in S$, then $||\boldsymbol{x}(t,x_0)||\leqslant\epsilon,\ \forall\ t\geqslant 0$. In addition, if the origin of the system converges from S in finite-time, meaning that for any $\boldsymbol{x}_0\in S$, there exists a finite-time $0\leqslant T<+\infty$ such that $\boldsymbol{x}(t,\boldsymbol{x}_0)=\mathbf{0};\ \forall\ t\geqslant T$, then the origin is considered to be finite-time stable within a predetermined time. The function $T(\boldsymbol{x}_0)=\inf\{T\geqslant 0:\boldsymbol{x}(t,\boldsymbol{x}_0)=\mathbf{0},\ \forall\ t\geqslant T\}$ represents the settling-time function for system (1). It provides the minimum time required for the system to reach the origin starting from the initial condition $\boldsymbol{x}_0$.

Lemma 1. *[26] If we have a positive definite function $V(t)$ satisfying*

$$\dot{V}(t)\leqslant -cV^{\beta}(t),\ \forall\ t\geqslant t_0;\ V(t_0)\geqslant 0$$

with constants $c>0$ and $\beta\in(0,1)$, then, for $t_0\leqslant t\leqslant t_1$,$V(t)$ will satisfy the following inequality

$$V^{1-\beta}(t)\leqslant V^{1-\beta}(t_0)-c(1-\beta)(t-t_0).$$

Additionally, $V(t)$ will be identically zero for all $t\geqslant t_1$, where t_1 is determined as

$$t_1=t_0+\frac{V^{1-\beta}(t_0)}{c(1-\beta)}. \tag{2}$$

Proof: We take a differential equation as:

$$\dot{X}(t)=-cX^{\eta}(t),\quad X(t_0)=V(t_0) \tag{3}$$

Although, the global Lipschitz condition is not met by the above differential equation, the unique solution to the above equation can be given as

$$X^{1-\eta}(t)=X^{1-\eta}(t_0)-c(1-\eta)(t-t_0) \tag{4}$$

and

$$X(t)\equiv 0,\quad \forall t\geqslant t_1. \tag{5}$$

It is direct to prove that $X(t)$ is differential for $t>t_0$. We get the following inequality by utilizing the comparison lemma [27]:

$$V^{1-\eta}(t)\leqslant V^{1-\eta}(t_0)-c(1-\eta)(t-t_0),t_0\leqslant t\leqslant t_1, \tag{6}$$

and

$$V(t)\equiv 0,\quad \forall t\geqslant t_1 \tag{7}$$

where t_1 is given by (2).

Lemma 2. *The inequality $(a_1+a_2+\cdots+a_n)^k\leqslant(a_1^k+a_2^k+\cdots+a_n^k)$ holds true, where a_i $(i=1,\cdots,n)$ and k are positive numbers.*

The proof of the above lemma is given in [28] and references therein.

III. PROBLEM FORMULATION

This section presents the general form of nonlinear systems used to represent the dynamics of both the master and slave systems. The master system is described using the following generalized model

$$\begin{aligned}\dot{x}_i&=x_{i+1},\quad i=1,2,\ldots,(n-1)\\ \dot{x}_n&=-\sum_{i=1}^{n}P_ix_i(t-\tau_{m_i})+f_n(x_1,\ldots,x_n),\end{aligned} \tag{8}$$

where $\boldsymbol{x}\in\mathbb{R}^n$ represents the state vector of the master system, and $P_i's$ are constants associated with the linear terms in the last equation of the system. The index i runs from 1 to n, and the linear terms are coupled with a constant time delay denoted by τ_{m_i}. The term $f_n(x_1,\ldots,x_n)$ represents the nonlinear function that characterizes the dynamics of the n^{th} state.

Similarly, the controlled slave system can be expressed as

$$\begin{aligned}\dot{y}_i&=y_{i+1},\quad i=1,2,\ldots,(n-1)\\ \dot{y}_n&=-\sum_{i=1}^{n}Q_iy_i(t-\tau_{s_i})+g_n(y_1,\ldots,y_n)+u,\end{aligned} \tag{9}$$

where $\boldsymbol{y}\in\mathbb{R}^n$ represents the state vector of the slave system and $Q_i's,\tau_{s_i},g_n(y_1,\ldots,y_n)$ are defined similarly as in the case of master system dynamics. From (8) and (9), the error dynamics can be written as

$$\begin{aligned}\dot{e}_i&=e_{i+1},\quad i=1,2,\ldots,(n-1)\\ \dot{e}_n&=-\sum_{i=1}^{n}Q_iy_i(t-\tau_{s_i})+\sum_{i=1}^{n}P_ix_i(t-\tau_{m_i})\\ &\quad+g_n(y_1,\ldots,y_n)-f_n(x_1,\ldots,x_n)+u,\end{aligned} \tag{10}$$

where $e_i=y_i-x_i,i=1,\ldots,n$ is the error between the states of the slave and master system dynamics.

Thus, the objective is to design the control function u such that $\lim_{t\to T}||\boldsymbol{y}-\boldsymbol{x}||=\lim_{t\to T}||\boldsymbol{e}||\to 0$.

IV. MAIN RESULTS

The theorem given below presents the finite-time synchronizing controller for (10) and the proof of the theorem describes the procedure to derive the expression of the finite-time synchronizing controller analytically.

Theorem 1. *For the error dynamics defined in (10), if the control input is selected as*

$$\begin{aligned}u=&\sum_{i=1}^{n}Q_iy_i(t-\tau_{s_i})-\sum_{i=1}^{n}P_ix_i(t-\tau_{m_i})-g_n(y_1,\ldots,y_n)\\ &+f_n(x_1,\ldots,x_n)+\dot{\alpha}_{n-1}-z_{n-2}-z_{n-1}^{\eta}\end{aligned} \tag{11}$$

where $z_i=e_{i+1}-\alpha_i,i=1,\ldots,(n-1)$ are the auxiliary variables, $\eta\in(0,1)$ and the virtual control inputs are defined as

$$\begin{aligned}\alpha_1&=-e_1^{\eta},\\ \alpha_i&=\dot{\alpha}_{i-1}-z_{i-2}-z_{i-1}^{\eta},i=2,\ldots,(n-1)\end{aligned} \tag{12}$$

where $z_0=e_1$, then all the states of the error dynamics are stabilized asymptotically in finite-time.

Proof: We use the traditional backstepping approach along with Lyapunov stability theorem to derive the finite-time synchronizing controller. Let us take the first subsystem of the error dynamics

$$\dot{e}_1 = e_2. \tag{13}$$

Selecting a Lyapunov function $V_1(e_1) = \frac{1}{2}e_1^2$ and differentiating it w.r.t. time gives $\dot{V}_1 = e_1 e_2$. Now, assuming $e_2 = \alpha_1 = -e_1^\eta$ as a virtual control input then $\dot{V}_1$ can be expressed as

$$\dot{V}_1 = -e_1^{\eta+1} = -(2)^{\frac{\eta+1}{2}}\left(\frac{1}{2}e_1^2\right)^{\frac{\eta+1}{2}}. \tag{14}$$

The above expression can be expressed as $\dot{V}_1 = -cV_1^\beta$ where $\beta = \frac{\eta+1}{2}$ and $c = 2^\beta$. By utilizing the Lemma 1, convergence of $e_1 \equiv 0$ in finite-time can be shown, thereby proving $\alpha_1 \equiv 0$. By taking an auxiliary variable, $z_1 = e_2 - \alpha_1$, the dynamics for the first two subsystems (e_1, z_1) is written as

$$\begin{aligned} \dot{e}_1 &= z_1 - e_1^\eta, \\ \dot{z}_1 &= e_3 - \dot{\alpha}_1. \end{aligned} \tag{15}$$

We chose another Lyapunov function $V_2(e_1, z_1) = \frac{1}{2}\left(e_1^2 + z_1^2\right)$, then $\dot{V}_2 = e_1(z_1 - e_1^\eta) + z_1(e_3 - \dot{\alpha}_1)$. Now, assuming $e_3 = \alpha_2 = \dot{\alpha}_1 - e_1 - z_1^\eta$ as another virtual control input and substituting it in the above expression, then $\dot{V}_2$ can be further expressed as

$$\begin{aligned} \dot{V}_2 &= -e_1^{\eta+1} - z_1^{\eta+1} \\ &= -(2)^{\frac{\eta+1}{2}}\left(\left(\frac{1}{2}e_1^2\right)^{\frac{\eta+1}{2}} + \left(\frac{1}{2}z_1^2\right)^{\frac{\eta+1}{2}}\right) \\ &\leqslant -(2)^{\frac{\eta+1}{2}}\left(\frac{1}{2}e_1^2 + \frac{1}{2}z_1^2\right)^{\frac{\eta+1}{2}} \leqslant -cV_2^\beta \end{aligned} \tag{16}$$

where $\beta = \frac{\eta+1}{2}$ and $c = 2^\beta$. Also, utilizing Lemma 1, convergence of $z_1 \equiv 0$ in finite-time can be shown. Since, it is assumed earlier that $z_1 = e_2 - \alpha_1$, consequently $e_2 \equiv 0$ as α_1 is a function of e_1 only and $e_1 \equiv 0$ has been proved earlier. Continuing in the similar manner, the subsystem dynamics till $(n-1)^{th}$ stage is written as

$$\begin{aligned} \dot{e}_1 &= z_1 - e_1^\eta, \\ \dot{z}_1 &= -e_1 + z_2 - z_1^\eta, \\ &\vdots \\ \dot{z}_{n-3} &= -z_{n-4} + z_{n-2} - z_{n-3}^\eta, \\ \dot{z}_{n-2} &= \dot{e}_{n-1} - \dot{\alpha}_{n-1} = e_n - \dot{\alpha}_{n-1}. \end{aligned} \tag{17}$$

Taking a Lyapunov function $V_{n-1}(e_1, z_1, \ldots, z_{n-2}) = \frac{1}{2}\left(e_1^2 + z_1^2 + \ldots, z_{n-2}^2\right)$ and computing its time derivative leads to the following expression

$$\begin{aligned} \dot{V}_{n-1} = {} & e_1(z_1 - e_1^\eta) + z_1(-e_1 + z_2 - z_1^\eta) + \cdots + \\ & z_{n-3}(-z_{n-4} + z_{n-2} - z_{n-3}^\eta) + z_{n-2}(e_n - \dot{\alpha}_{n-1}). \end{aligned} \tag{18}$$

Assuming $e_n = \dot{\alpha}_{n-1} - z_{n-3} - z_{n-2}^\eta$ and substituting it in the above equation then (18) is rewritten as

$$\begin{aligned} \dot{V}_{n-1} = {} & -e_1^{\eta+1} - z_1^{\eta+1} - \cdots - z_{n-3}^{\eta+1} - z_{n-2}^{\eta+1} \\ = {} & -(2)^{\frac{\eta+1}{2}}\left(\left(\frac{1}{2}e_1^2\right)^{\frac{\eta+1}{2}} + \left(\frac{1}{2}z_1^2\right)^{\frac{\eta+1}{2}} + \cdots + \right. \\ & \left. \left(\frac{1}{2}z_{n-3}^2\right)^{\frac{\eta+1}{2}} + \left(\frac{1}{2}z_{n-2}^2\right)^{\frac{\eta+1}{2}}\right) \\ \leqslant {} & -(2)^{\frac{\eta+1}{2}}\left(\frac{1}{2}e_1^2 + \frac{1}{2}z_1^2 + \cdots + \frac{1}{2}z_{n-3}^2 + \frac{1}{2}z_{n-2}^2\right)^{\frac{\eta+1}{2}} \\ \leqslant {} & -cV_{n-1}^\beta. \end{aligned} \tag{19}$$

where $\beta = \frac{\eta+1}{2}$ and $c = 2^\beta$. Again, utilizing Lemma 1, convergence of $z_{n-2} \equiv 0$ in finite-time can be achieved, consequently proving $e_{n-1} \equiv 0$. Finally, considering the subsystem dynamics for the n^{th} stage

$$\begin{aligned} \dot{e}_1 &= z_1 - e_1^\eta, \\ \dot{z}_1 &= -e_1 + z_2 - z_1^\eta, \\ &\vdots \\ \dot{z}_{n-2} &= -z_{n-3} + z_{n-1} - z_{n-2}^\eta, \\ \dot{z}_{n-1} &= -\sum_{i=1}^{n} Q_i y_i(t - \tau_{s_i}) + \sum_{i=1}^{n} P_i x_i(t - \tau_{m_i}) \\ &\quad + g_n(y_1, \ldots, y_n) - f_n(x_1, \ldots, x_n) + u - \dot{\alpha}_{n-1}. \end{aligned} \tag{20}$$

We take a Lyapunov function $V_n(e_1, z_1, \ldots, z_{n-1}) = V_{n-1} + \frac{1}{2}z_{n-1}^2$. Taking the time derivative of this function

$$\begin{aligned} \dot{V}_n = \dot{V}_{n-1} + z_{n-1}\Bigg(& -\sum_{i=1}^{n} Q_i y_i(t - \tau_{s_i}) + \\ & \sum_{i=1}^{n} P_i x_i(t - \tau_{m_i}) + g_n(y_1, \ldots, y_n) - \\ & f_n(x_1, \ldots, x_n) + u - \dot{\alpha}_{n-1}\Bigg) \end{aligned} \tag{21}$$

Substituting the controller expression defined in (11), then the above equation is expressed in the following form

$$\begin{aligned} \dot{V}_n \leqslant {} & -e_1^{\eta+1} - z_1^{\eta+1} - \cdots - z_{n-2}^{\eta+1} - z_{n-1}^{\eta+1} \\ \leqslant {} & -(2)^{\frac{\eta+1}{2}}\left(\left(\frac{1}{2}e_1^2\right)^{\frac{\eta+1}{2}} + \left(\frac{1}{2}z_1^2\right)^{\frac{\eta+1}{2}} + \cdots + \right. \\ & \left. \left(\frac{1}{2}z_{n-2}^2\right)^{\frac{\eta+1}{2}} + \left(\frac{1}{2}z_{n-1}^2\right)^{\frac{\eta+1}{2}}\right) \\ \leqslant {} & -(2)^{\frac{\eta+1}{2}}\left(\frac{1}{2}e_1^2 + \frac{1}{2}z_1^2 + \cdots + \frac{1}{2}z_{n-1}^2\right) \\ \leqslant {} & -cV_n^\beta \end{aligned} \tag{22}$$

where $\beta = \frac{\eta+1}{2}$ and $c = 2^\beta$. Utilizing Lemma 1, convergence of $z_{n-1} \equiv 0$ in finite-time is achieved. Therefore, $e_n \equiv 0$. As all the states of the error dynamics defined in (10) are finite-time stabilized, therefore, it is concluded that after a finite-time $T > 0, \boldsymbol{e}_i \to 0, (i = 1, 2, \ldots, n)$, thereby implying the synchronization between the states of the master and the slave system in finite-time.

V. NUMERICAL SIMULATION

In this section, the validation of the methodology given in previous section is done by presenting the simulation result of the synchronization between the master and slave systems of Genesio-Tesi chaotic system and Sprott Circuit in finite-time.

A. Finite-time synchronization of master-slave system of Genesio-Tesi chaotic system

The dynamical description of the time delayed master system of Genesio-Tesi chaotic system is described as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -cx_1(t-\tau_{m_1}) - bx_2(t-\tau_{m_2}) - \\ &\quad ax_3(t-\tau_{m_3}) + x_1^2, \end{aligned} \tag{23}$$

where $a = 1.2, b = 2.92, c = 6$ are the parameter values. $\tau_{m_1}, \tau_{m_2}, \tau_{m_3}$ are the time delays associated in the last equation of the system dynamics. For the initial state values $\boldsymbol{x}(0) = [1.5, -0.15, 0.1]^T$, the chaotic behavior of its states w.r.t. time is illustrated in Fig. 1(a)-(c). For $n = 3$, the above system dynamics fits in the structure of the generalized dynamical system defined in (8), if we consider $P_1 = -c$, $P_2 = -b$, $P_3 = -a$ and $f_3(x_1) = x_1^2$. The differential dynamic description of the controlled slave system is given as

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -cy_1(t-\tau_{s_1}) - by_2(t-\tau_{s_2}) - ay_3(t-\tau_{s_3}) + \\ &\quad y_1^2 + u, \end{aligned} \tag{24}$$

where $\tau_{s_1}, \tau_{s_2}, \tau_{s_3}$ are the time delays associated with the slave system dynamics. To achieve the synchronization between the master and slave states, the expression for the finite-time synchronizing controller as defined in (11) is written as

$$\begin{aligned} u &= cy_1(t-\tau_{s_1}) + by_2(t-\tau_{s_2}) + ay_3(t-\tau_{s_3}) \\ &\quad - cx_1(t-\tau_{m_1}) - bx_2(t-\tau_{m_2}) - ax_3(t-\tau_{m_3}) \\ &\quad - y_1^2 + x_1^2 + \dot{\alpha}_2 - z_1 - z_2^\eta \end{aligned} \tag{25}$$

where the delays associated with the master system are taken as $\tau_{m_1} = \tau_{m_2} = \tau_{m_3} = 0.02$ whereas the delays associated with the slave system are taken as $\tau_{s_1} = 0.04, \tau_{s_1} = 0.01, \tau_{s_1} = 0.03$. The auxiliary variables are defined as $z_1 = e_2 - \alpha_1$, $z_2 = e_3 - \alpha_2$ whereas the virtual controllers defined by (12) is given as $\alpha_1 = -e_1^\eta$, $\alpha_2 = \dot{\alpha}_1 - e_1 - z_1^\eta$ where $\eta = 0.8$. Figs 2(a)-(c) illustrates the synchronization of the master-slave system where the states of the slave system are tracking the states of the master system in finite-time. Fig. 2d shows the synchronization error between the pair of corresponding states of the master and slave system. From the figure, it is concluded that the synchronizing error is converging to zero in finite-time. Fig. 2e shows the variation of the control input over the simulation duration of 20 seconds.

B. Finite-time synchronization of master-slave system of Sprott Circuit

The dynamical description of the time delayed master system of Sprott circuit is described as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -x_2(t-\tau_{m_2}) - ax_3(t-\tau_{m_3}) - \\ &\quad 1.2x_1(t-\tau_{m_1}) + 2\,\mathrm{sgn}(x_1), \end{aligned} \tag{26}$$

where $a = 0.6$ is the parameter value. $\tau_{m_1}, \tau_{m_2}, \tau_{m_3}$ are the time delays. For the initial state values $\boldsymbol{x}(0) = [-0.5, -0.5, 0.5]^T$, the chaotic behavior of its states w.r.t. time is illustrated in Fig. 3(a)-(c). For $n = 3$, the above system dynamics fits in the structure of the generalized dynamical system defined in (8), if we consider $P_1 = -1.2$, $P_2 = 1$, $P_3 = -a$ and $f_3(x_1) = 2\,\mathrm{sgn}(x_1)$. The differential dynamic description of the controlled slave system is given as

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -y_2(t-\tau_{s_2}) - ay_3(t-\tau_{s_3}) - 1.2y_1(t-\tau_{s_1}) \\ &\quad + 2\,\mathrm{sgn}(y_1) + u, \end{aligned} \tag{27}$$

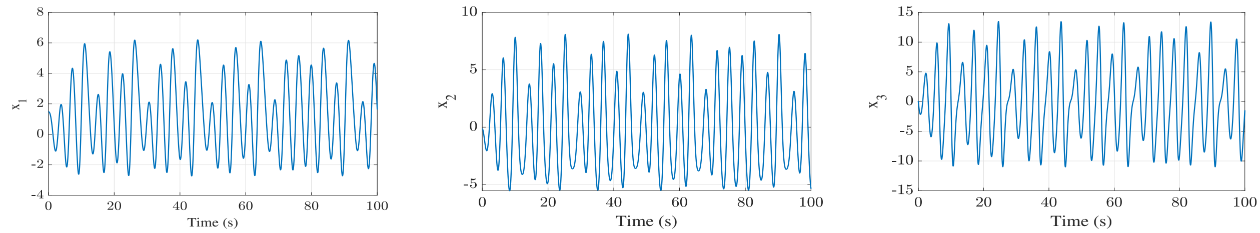
where $\tau_{s_1}, \tau_{s_2}, \tau_{s_3}$ are the time delays associated with the slave system dynamics. The time delays associated with the master system are taken as $\tau_{m_1} = \tau_{m_2} = \tau_{m_3} = 0.1$ and the time delays associated with the slave system is taken as $\tau_{s_1} = 0.4, \tau_{s_2} = 0.2, \tau_{s_1} = 0.3$. As defined in (11), the expression for the finite-time synchronization controller is given by

$$\begin{aligned} u &= y_2(t-\tau_{s_2}) + ay_3(t-\tau_{s_3}) + 1.2(t-\tau_{s_1}) - \\ &\quad 2\,\mathrm{sgn}(y_1) - x_2(t-\tau_{m_2}) - ax_3(t-\tau_{m_3}) - \\ &\quad 1.2x_1(t-\tau_{m_1}) + 2\,\mathrm{sgn}(x_1) - z_1 - z_2^\eta \end{aligned} \tag{28}$$

where the expressions for the auxiliary variables and the virtual control inputs are similar to the previous example. Figs 4(a)-(c) illustrates the synchronization of the master-slave system where the states of the slave system are tracking the states of the master system in finite-time. Fig. 4d shows the synchronization error between the pair of corresponding states of the master and slave system. From the figure, it is concluded that the synchronizing error is converging to zero in finite-time. Fig. 4e shows the variation of the control input over the simulation duration of 20 seconds.

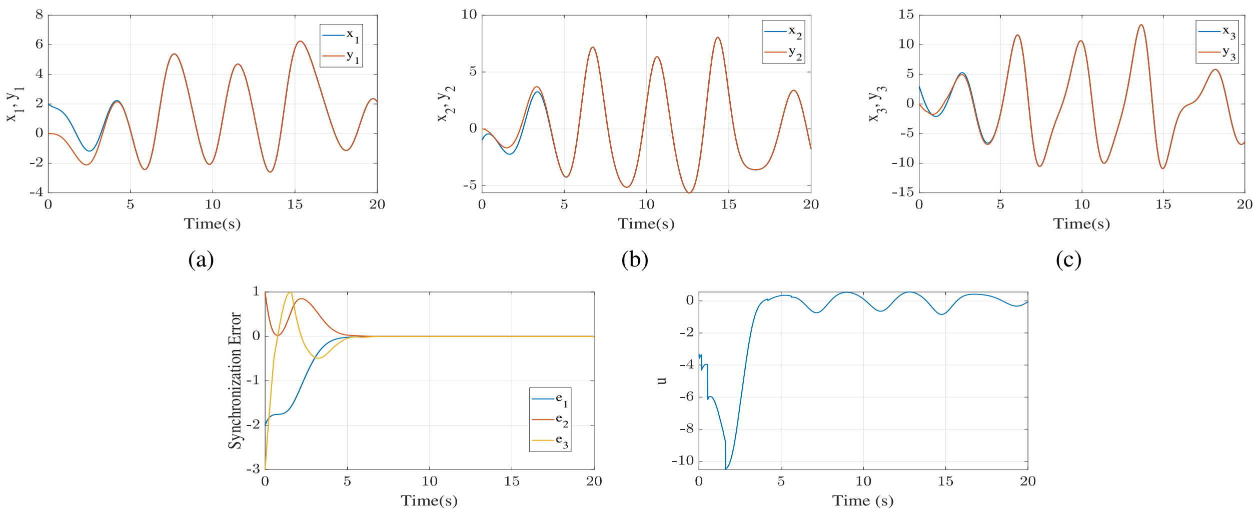
VI. CONCLUSION

In this study, we proposed a control framework to achieve finite-time synchronization for a specific class of nonlinear systems affected by constant time delays. Utilizing back-stepping technique, we derived a finite-time synchronizing controller that guarantees the synchronization error between



(a) (b) (c)

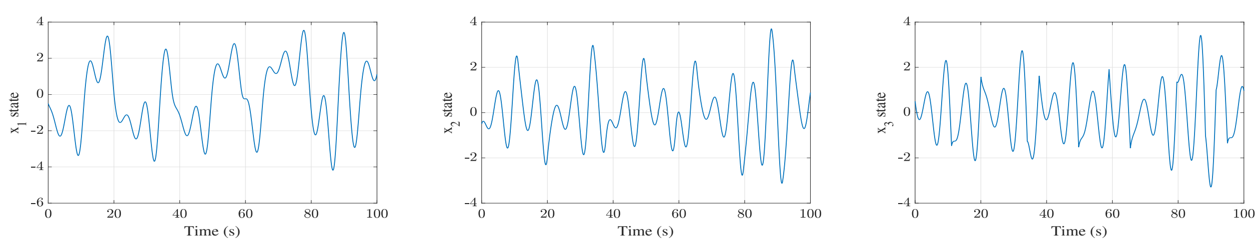
Fig. 1: Chaotic behavior of the state trajectories of the uncontrolled Genesio-Tesi chaotic system (a) Variation of the first state w.r.t. time (b) Variation of the second state w.r.t. time (c) Variation of the third state w.r.t. time.



(a) (b) (c)

(d) (e)

Fig. 2: Synchronization of the states between the master and slave system of Genesio-Tesi system having constant time delays (a) Synchronization between the first states of the master-slave system (b) Synchronization between the second states of the master-slave system (c) Synchronization between the third states of the master-slave system (d) Variation of the convergence of the synchronization error w.r.t. time (e) Variation of the control input.



(a) (b) (c)

Fig. 3: Chaotic behavior of the state trajectories of the uncontrolled Sprott Circuit system (a) Variation of the first state w.r.t. time (b) Variation of the second state w.r.t. time (c) Variation of the third state w.r.t. time.

the master and slave systems converges to zero within a finite time frame. The theoretical analysis, based on Lypunov-based analysis and the backstepping method, confirms that the error dynamics are stabilized in finite-time. The methodology has been validated using the Genesio-Tesi chaotic system and Sprott Circuits system. Simulation results demonstrated the good performance of the proposed controller. While the finite-time controller successfully handles the constant time delays, certain challenges such as uncertain, time-varying delays along with the unmatched uncertainties or finite-time synchronization of the states between heterogeneous systems having time delays are reserved for future investigation.

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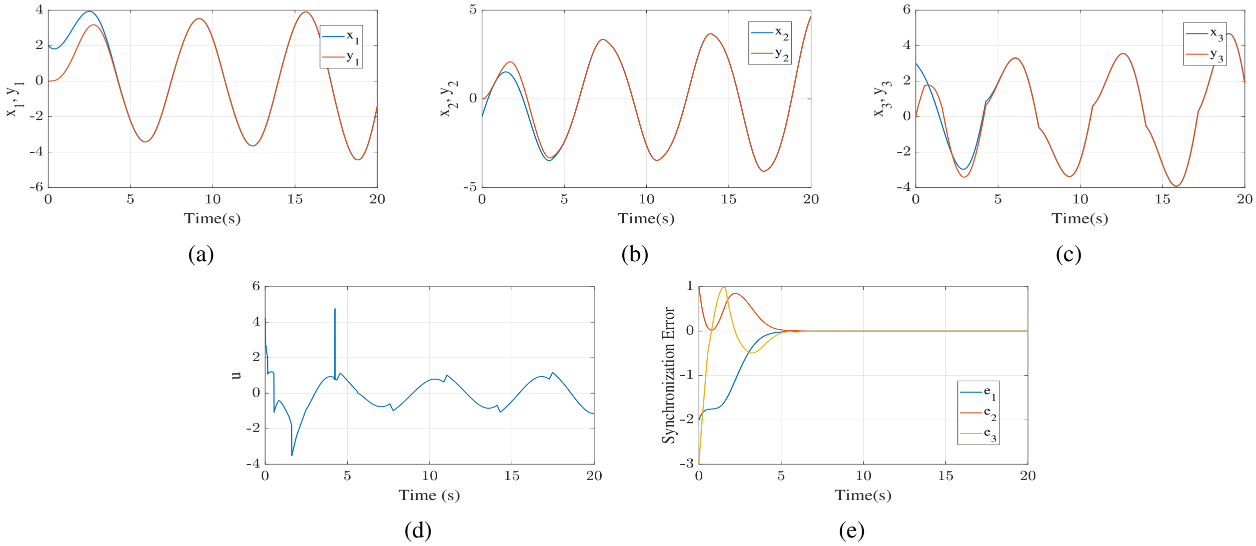

Fig. 4: Synchronization of the states between the master and slave system of Sprott Circuit system having constant time delays (a) Synchronization between the first states of the master-slave system (b) Synchronization between the second states of the master-slave system (c) Synchronization between the third states of the master-slave system (d) Variation of the control input (e) Variation of the convergence of the synchronization error w.r.t. time.

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