Fault Detection for Wastewater Treatment Plants Based on H_-/L_∞ Observer

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Abstract—This paper presents an H_-/L_∞ fault detection observer for the activated sludge process in wastewater treatment plants. First, the nonlinear mass balance model of the activated sludge process is transformed into a Takagi–Sugeno (T-S) fuzzy system. Due to the presence of unmeasured premise variables, the errors resulting from these variables and external disturbances are treated as a whole in the observer design. Subsequently, an observer is developed to achieve high fault sensitivity while effectively attenuating disturbances. Finally, the effectiveness of the proposed method is demonstrated through simulations and compared with a conventional L_∞ observer.

I. INTRODUCTION

Fault detection in wastewater treatment plants is vital. Failures lead to environmental operational problems. The key activated-sludge process is vulnerable to actuator, pump issues, and influent quality shifts. Timely detection ensures water quality and prevents costly repairs shutdowns [1].

Traditional fault detection methods often struggled with uncertainty and external disturbances, especially in nonlinear systems [2]. To address these challenges, robust fault detection observers, such as H_-/L_∞ observers and H_-/H_∞ observers [3], were proposed as effective tools for handling uncertainties and improving fault detection accuracy. The L_∞ technique exhibited excellent performance in the presence of disturbances, sensor noise, and model uncertainties, making them suitable for dynamic and complex processes. A discrete-time H_-/L_∞ fault diagnosis observer was designed in [4]. [5] developed a T-S fuzzy observer with unknown inputs, and [6] proposed a method for designing an observer for T-S fuzzy systems with measurable premise variables. In addition, [7] proposed an observer design method for a class of Lipschitz nonlinear systems with parameter uncertainties, and [8] constructed a prescribed-time observer for LPV systems. The Kalman filter was also widely used in the field of fault detection [16]. In addition to the L_∞ technique for handling uncertainties, set-membership estimation was also commonly used in fault diagnosis. Ellipsoidal analysis was employed in [17] to perform state interval estimation for T-S fuzzy systems.

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This paper presents an H_-/L_∞ observer for detecting faults in the activated sludge process of wastewater treatment plants. The nonlinear activated sludge mass balance model is transfrormed into a T-S fuzzy system, capturing the system's nonlinearities and uncertainties. By treating unmeasured premise variables and disturbances as a unified error term, the observer effectively detects faults while maintaining robustness against uncertainties.

The proposed method's effectiveness is confirmed through simulations, where it is compared to conventional L_∞ observers, showing superior fault detection capabilities.

The remainder of the paper is structured as follows. Section II presents the system model and the formulation of the Takagi-Sugeno fuzzy system. Section III details the design of the H_-/L_∞ fault detection observer and the optimization of the error bounds. Section IV shows the simulation results and contrasts the performance of the proposed observer with that of the L_∞ observer. Finally, Section V concludes the paper by reviewing the research findings.

II. SYSTEM DESCRIPTION

This section transforms the proposed nonlinear system into a Takagi-Sugeno fuzzy system, consisting of local linear sub-models, using the T-S fuzzy method.

According to the wastewater treatment model described in [9], oxygen is first introduced into the wastewater in the aeration tank, where it is used by the aerobic microorganisms in the activated sludge to decompose organic matter. Meanwhile, a portion of the activated sludge is continuously recirculated from the sedimentation tank to maintain the stability of the sludge biomass in the system. The dynamic behavior of the system can be described by the following equations, the definitions of the symbols are provided in Table I.

$$\begin{cases} \dfrac{dX_b}{dt} = \vartheta(t)X_b(t) - D(1+q_r)X_b(t) + q_r D X_{rb}(t) \\ \dfrac{dX_{rb}}{dt} = D(1+q_r)X_b(t) - D(\kappa+q_r)X_{rb}(t) \\ \dfrac{dS}{dt} = -\dfrac{1}{\varrho}\vartheta(t)X_b(t) - (1+q_r)DS(t) + \\ \qquad DC_{s_{in}}(t) \\ \dfrac{dC_{do}}{dt} = -\dfrac{\mathfrak{K}_0}{\varrho}\vartheta(t)X_b(t) - D(1+q_r)C_{do}(t) + \\ \qquad DC_{0_{in}}(t) + K_{La}(C_{0_{max}}(t) - C_{do}(t)) \end{cases} \tag{1}$$

TABLE I: NOTATION AND DEFINITIONS

Symbol	Definition
$X_b(t)$	Biomass concentration
$X_{rb}(t)$	Return biomass concentration
$S(t)$	Substrate concentration
$C_{do}(t)$	Dissolved oxygen concentration
ϱ	Constant yield coefficient
$\mathfrak{K}_0$	Constant coefficient
$C_{s_{in}}(t)$	Influent substrate concentration
$C_{0_{in}}(t)$	Influent dissolved oxygen concentration
$C_{0_{max}}$	Maximum dissolved oxygen concentration
q_r	Ratio of the recycled flow to the influent flow
κ	Ratio of the wasted flow to the influent flow
ϑ_{max}	Maximum specific growth rate
$\mathfrak{K}_s$	Saturation factor
$\mathfrak{K}_a$	Affinity factor
D	Dilution rate
K_{La}	Aeration flow rate

where $\vartheta(t)$ evolves according to the following adaptive law:

$$\vartheta(t) = \vartheta_{max} \frac{S(t)}{\mathfrak{K}_a + S(t)} \frac{C_{do}(t)}{\mathfrak{K}_s + C_{do}(t)} \tag{2}$$

Under the assumption that only the dissolved oxygen concentration is considered as the measured variable, and both faults and external disturbances are neglected, system (1) can be transformed into a nonlinear system.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3}$$

where

$$A = \begin{bmatrix} \vartheta(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\varrho}\vartheta(t) & 0 & 0 & 0 \\ -\frac{\mathfrak{K}_0}{\varrho}\vartheta(t) & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -(1+q_r)X_b(t) + q_r X_{rb}(t) & 0 \\ (1+q_r)X_b(t) - (\kappa + q_r)X_{rb}(t) & 0 \\ C_{s_{in}} - (1+q_r)S(t) & 0 \\ C_{0_{in}} - (1+q_r)C_{do}(t) & C_{0_{max}} - C_{do}(t) \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

The system described by (3) is nonlinear. Before performing T-S fuzzy linearization, According to [9] the following six antecedent variables are selected, which result in $2^6 = 64$ fuzzy subsystems.

$$\begin{cases} \rho_1(t) = \vartheta(t) - S(t) \\ \rho_2(t) = X_b(t) \\ \rho_3(t) = X_{rb}(t) \\ \rho_4(t) = C_{do}(t) \\ \rho_5(t) = \vartheta_{max} \dfrac{S(t)X_b(t)}{(\mathfrak{K}_s + S(t)(\mathfrak{K}_a + C_{do}(t))} \\ \rho_6(t) = S(t) \end{cases} \tag{4}$$

Then, system (3) can be transformed into the following form.

$$\begin{cases} \dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where

$$A(\rho) = \begin{bmatrix} \rho_1 & 0 & \rho_2 & 0 \\ 0 & -\rho_4 & 0 & \rho_3 \\ 0 & 0 & -\frac{1}{\varrho}\rho_5 & 0 \\ -\rho_3 & \rho_2 & 0 & -\frac{\mathfrak{K}_0}{\varrho}\rho_5 \end{bmatrix}$$

$$B(\rho) = \begin{bmatrix} -(1+q_r)\rho_2 + q_r\rho_3 & 0 \\ (1+q_r)\rho_2 - (\kappa + q_r)\rho_3 & 0 \\ C_{s_{in}} - (1+q_r)\rho_6 & 0 \\ C_{0_{in}} - (1+q_r\rho_4) & C_{0_{max}} - \rho_4 \end{bmatrix}$$

Let $\rho_{j,\max}$ and $\rho_{j,\min}$ denote the maximum and minimum values of ρ_j, respectively, where j is a positive integer ranging from 1 to 6. The fuzzy membership function can then be constructed using the following method:

$$\begin{aligned} \mathcal{F}_{j,\max}(\rho_j(t)) &= \frac{\rho_j(t) - \rho_{j,\min}}{\rho_{j,\max} - \rho_{j,\min}} \\ \mathcal{F}_{j,\min}(\rho_j(t)) &= \frac{\rho_{j,\max} - \rho_j(t)}{\rho_{j,\max} - \rho_{j,\min}} \end{aligned} \tag{6}$$

This paper considers the system to be influenced by an unknown yet bounded disturbance $d(t)$ and noise $g(t)$ during operation, ensuring that

$$\|d(t)\| \le \bar{d}, \quad \|g(t)\| \le \bar{g}. \tag{7}$$

where $\bar{d}$ and $\bar{g}$ denote the upper bounds of the disturbance and the noise, respectively.

According to [11], the following equations are now defined to simplify subsequent calculations.

$$\mathcal{R}(\rho) = \sum_{i=1}^{n_r} \epsilon_i(\rho(t))\mathcal{R}_i \tag{8}$$

where $\mathcal{R}$ is an arbitrary matrix.

Apply fuzzy rules and fuzzy membership functions, and consider disturbances and actuator fault to give the following fuzzy system.

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n_r} \epsilon_i(\rho)(A_i x(t) + B_i u(t) + D_x d(t) + Ff(t)) \\ y(t) = Cx(t) + D_y g(t) \end{cases} \tag{9}$$

where $\epsilon_i(\rho)$, $i = 1, \ldots, 64$, is the weighting function obtained by integrating the multiplication part of each affiliation function. $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $F \in \mathbb{R}^{n_x \times n_f}$, $C \in \mathbb{R}^{n_y \times n_x}$, $D_x \in \mathbb{R}^{n_x \times n_d}$ and $D_y \in \mathbb{R}^{n_y \times n_g}$ are system matrices. $d(t) \in \mathbb{R}^{n_d}$ and $g(t) \in \mathbb{R}^{n_g}$represents the process unknown disturbance, $f(t) \in \mathbb{R}^{n_f}$ represents the actuator.

After the above transformation, system (9) can be simplified into the following form:

$$\begin{cases} \dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) + D_x d(t) + Ff(t) \\ y(t) = Cx(t) + D_y g(t) \end{cases} \tag{10}$$

The following observer structure is proposed:

$$\begin{cases} \dot{\hat{x}}(t) = A(\hat{\rho})\hat{x}(t) + B(\hat{\rho})u(t) + L(\hat{\rho})(y(t) - C\hat{x}(t)) \\ r(t) = y(t) - C\hat{x}(t) \end{cases} \tag{11}$$

where $\hat{x}(t) \in \mathbb{R}^{n_x}$ is the state estimation vector, $r(t) \in \mathbb{R}^{n_y}$ is the residual vector used to detect fault, and $L(\hat{\rho}) \in \mathbb{R}^{n_x \times n_y}$ is the gain matrix to be designed.

The estimation error of the observer system is defined as.

$$e(t) = x(t) - \hat{x}(t) \tag{12}$$

Then, the following dynamic error dynamic holds.

$$\begin{cases} \dot{e} = (A(\hat{\rho}) - L(\hat{\rho})C)e(t) + \phi(t) + Ff(t) \\ r(t) = y(t) - C\hat{x}(t) \end{cases} \tag{13}$$

According to [12], the errors and disturbances generated by the antecedent variables can be integrated and treated as a single term:

$$\begin{aligned} \phi(t) = &\sum_{i=1}^{n_r} (\epsilon_i(\rho(t)) - \epsilon_i(\hat{\rho}(t))(A_i x(t) + B_i u(t)) \\ &- L(\hat{\epsilon}) D_y g(t) + D_x d(t) \end{aligned} \tag{14}$$

It is assumed that $\epsilon(\hat{\rho}(t))$ is bounded. Since the state of the control system is also bounded, the entire $\phi(t)$ term is therefore bounded.

The main objective of this paper is to design the observer (11) to satisfy the H_-/L_∞ performance criterion proposed in [13].

III. MAIN RESULTS

III-A. H_-/L_∞ fault detection observer design

This section presents the design method of the H_-/L_∞ fault detection observer for the T-S fuzzy system.

Theorem 1: For system (9), given scalars $\beta > 0$, γ_1, $\gamma_2 > 0$, for any $i = 1, 2, 3, \ldots, 64$, if there exist scalars $\alpha > 0$ and matrices $P_d(\hat{\rho}) = P_d^T(\hat{\rho}) > 0$, $P_f(\hat{\rho}) = P_f^T(\hat{\rho}) > 0 \in \mathbb{R}^{n_x \times n_x}$ and $E = E^T \in \mathbb{R}^{n_x \times n_x}$ such that the following inequalities hold:

$$\begin{bmatrix} \Delta_1 & P_d(\hat{\rho}) \\ P_d(\hat{\rho}) & -\alpha I \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} C^T C - \gamma_1^2 P_{di} & C^T D_y \\ D_y^T C & -\gamma_2^2 I + D_y^T D_y \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_3 & \beta^2 I - F^T E F \end{bmatrix} < 0 \tag{17}$$

where

$$\begin{cases} \Delta_1 = (A(\hat{\rho}) - L(\hat{\rho})C)^T P_d(\hat{\rho}) + P_d(\hat{\rho})(A(\hat{\rho}) - L(\hat{\rho})C) \\ \qquad + \alpha P_d(\hat{\rho}) \\ \Xi_1 = (A(\hat{\rho}) - L(\hat{\rho})C)^T P_f(\hat{\rho}) + P_f(\hat{\rho})(A(\hat{\rho}) - L(\hat{\rho})C) \\ \qquad - \omega_1\omega_2 E - j\omega_0 E(A(\hat{\rho}) - L(\hat{\rho})C) \\ \qquad + (A(\hat{\rho}) - L(\hat{\rho})C)^T j\omega_0 E - C^T C - \\ \quad (A(\hat{\rho}) - L(\hat{h})C)^T E(A(\hat{\rho}) - L(\hat{\rho})C) \\ \Xi_2 = P_f(\hat{\rho})F - (A(\hat{\rho}) - L(\hat{\rho})C)EF + j\omega_0 EF \\ \Xi_3 = F^T P_f(\hat{\rho}) - F^T E(A(\hat{\rho}) - L(\hat{\rho})C) - F^T j\omega_0 E \end{cases}$$

Therefore, system (11) is an observer with H_-/L_∞ performance that satisfies the following performance indices, where $V_+(0)$ is the initial value of the Lyapunov function.

$$||r(t)|| \le \sqrt{\gamma_1 V_+(0) e^{-\alpha t} + (\gamma_1^2 + \gamma_2^2)||\phi(t)||_\infty^2} \tag{18}$$

$$\int_0^\infty \left[\beta^2 f^T(t) f(t) - r^T(t) r(t)\right] dt < 0 \tag{19}$$

Proof:

(1) Stability and L_∞ disturbance attenuation condition.

When considering only the influence of disturbances, let $f(t) = 0$, and the dynamic error system becomes:

$$\begin{cases} \dot{e} = (A(\hat{\rho}) - L(\hat{\rho})C)e(t) + \phi(t) \\ r(t) = y(t) - C\hat{x} \end{cases} \tag{20}$$

For (20), choose the Lyapunov function as:

$$V_+ = e^T(t) P_d(\hat{\rho}) e(t) \tag{21}$$

Then, the derivative of the Lyapunov function is:

$$\begin{aligned} \dot{V}_+ = &e(t)^T((A(\hat{\rho}) - L(\hat{\rho})C)^T P_d(\hat{\rho}) + P_d(\hat{\rho})(A(\hat{\rho}) \\ &- L(\hat{\rho})C))e(t) + \phi^T(t) P_d(\hat{\rho}) e(t) + e^T(t) P_d(\hat{\rho}) \phi(t) \end{aligned} \tag{22}$$

Let

$$\zeta = \begin{bmatrix} e^T(t) & \phi^T(t) \end{bmatrix} \tag{23}$$

$$\gamma_\Sigma = \begin{bmatrix} \mathbb{A}_d^T P_d(\hat{\rho}) + P_d(\hat{\rho})\mathbb{A}_d & P_d(\hat{\rho}) \\ P_d(\hat{\rho}) & P_d(\hat{\rho}) \end{bmatrix} \tag{24}$$

where

$$\mathbb{A}_d = (A(\hat{\rho}) - L(\hat{\rho})C) \tag{25}$$

It can be obtained

$$\dot{V}_+ = \zeta \gamma_\Sigma \zeta^T \tag{26}$$

According to [14], if the following conditions are satisfied

$$\dot{V}_+ \le -\alpha V_+ + \alpha \phi^T(t)\phi(t) \tag{27}$$

where $\alpha > 0$. According to the definition of the L_∞ norm, it follows that

$$V_+(t) \le \alpha V_+(t) + \alpha \|\phi(t)\|_\infty^2 \tag{28}$$

then

$$\begin{aligned} V_+ &\le (\alpha||\phi(t)||_\infty^2 e^{\alpha t} + V_+(0))e^{-\alpha t} \\ &\le V_+(0)e^{-\alpha t} + ||\phi(t)||_\infty^2 \end{aligned} \tag{29}$$

Substitute (20) and (21) into (27) and simplify to obtain (15).

Additionally, from inequality (16), we obtain:

$$\begin{bmatrix} e(t) \\ \phi(t) \end{bmatrix}^T \begin{bmatrix} C^T C - \gamma_1^2 P_{di} & C^T D_y \\ D_y^T C & -\gamma_2^2 I + D_y^T D_y \end{bmatrix} \begin{bmatrix} e(t) \\ \phi(t) \end{bmatrix} < 0 \tag{30}$$

Equivalent to

$$r^T(t) r(t) - \gamma_1 V_+(t) - \gamma_2^2 \phi^T(t)\phi(t) < 0 \tag{31}$$

Substitute equation (29) into equation (31), we get (18).

(2) H_- fault sensitivity condition.

Consider that the disturbance $\phi(t)=0$, then the error system becomes:

$$\begin{cases}\dot{e}=(A(\hat{\rho})-L(\hat{\rho})C)e(t)+Ff(t)\\ r(t)=y(t)-C\hat{x}(t)\end{cases} \tag{32}$$

For (32), chdsoose the Lyapunov function as:

$$V_-=e^T(t)P_f(\hat{\rho})e(t) \tag{33}$$

The derivative of the Lyapunov function is obtained as:

$$\begin{aligned}\dot{V}_- &= e^T(t)((A(\hat{\rho})-L(\hat{\rho})C)^T P_f(\hat{\rho})+P_f(\hat{\rho})(A(\hat{\rho})\\ &-L(\hat{\rho})C))e(t)-f^T(t)F^T P_f(\hat{\rho})e(t)\\ &-e^T(t)P_f(\hat{\rho})Ff(t))\end{aligned} \tag{34}$$

Introduce the evaluation index $\mathfrak{J}_-$:

$$\begin{aligned}\mathfrak{J}\_ = \int_0^\infty &\{\dot{V}_f+[\beta^2 f^T(t)f(t)-r^T(t)r(t)]\\ &-tr[\ \mathrm{He}(\omega_1 e(t)+j\dot{e}(t))(\omega_2 e(t)+j\dot{e}(t))^*E]\ \}\end{aligned} \tag{35}$$

where S is a symmetric matrix satisfying $E^T=E$, ω_1 and ω_2 are known constants, and β is a given constant.

Under zero initial conditions, if $\mathfrak{J}_-<0$ holds, then we have:

$$\begin{aligned}\mathfrak{J}\_ = \int_0^\infty &\{[\beta^2 f^T(t)f(t)-r^T(t)r(t)]\ -\\ tr[\ \mathrm{He}&(\omega_1 e(t)+j\dot{e}(t))(\omega_2 e(t)+j\dot{e}(t))^*E]\ \}<0\end{aligned} \tag{36}$$

Let

$$\mathcal{Z}=\int_0^\infty(\omega_1 e(t)+j\dot{e}(t))(\omega_2 e(t)+j\dot{e}(t))^* dt \tag{37}$$

Assuming that the actuator fault occurs in the frequency domain that satisfies the following equation:

$$\int_0^\infty \tau(\omega_1 e(t)+\mathrm{j}\dot{e}(t))(\omega_2 e(t)+\mathrm{j}\dot{e}(t))^*\mathrm{dt}\leqslant 0,$$

Thus $\tau\mathcal{Z}\le 0$. Therefore, when $\tau E\ge 0$ holds, the inequality $\mathrm{tr}(\mathcal{Z}E)\le 0$. Since matrix E is symmetric, we have:

$$tr(\mathcal{Z}E)=\mathrm{tr}(\mathcal{Z}^*E)=\mathrm{tr}(E\mathcal{Z}^*)\le 0$$

That is,

$$tr[\ \mathrm{He}(\omega_1 e(t)+j\dot{e}(t))(\omega_2 e(t)+j\dot{e}(t))^*E]\le 0 \tag{38}$$

Therefore, from (36), we can deduce:

$$\int_0^\infty\left\{[\beta^2 f^T(t)f(t)-r^T(t)r(t)]\right\}<\mathrm{tr}\left[\mathbf{He}(\mathcal{Z})E\right]\le 0 \tag{39}$$

That is, when $\mathfrak{J}_-<0$ holds,the error system satisfies the finite-frequency $H\_$ fault sensitivity condition.

Now compute $\mathrm{tr}[\mathbf{He}(\mathcal{Z})E]$, and substitute (34), which gives:

$$\begin{aligned}&tr[\ \mathrm{He}(\omega_1 e(t)+j\dot{e}(t))(\omega_2 e(t)+j\dot{e}(t))^*E]=\\ &e(t)^T\omega_1\omega_2 Ee(t)+(A(\hat{\rho})-L(\hat{\rho})C)e(t)+Ff(t))^T E\\ &\times((A(\hat{\rho})-L(\hat{\rho})C)e(t)+Ff(t))+e(t)^T j\omega_0 E\\ &\times((A(\hat{\rho})-L(\hat{\rho})C)e(t)+Ff(t))-((A(\hat{\rho})-L(\hat{\rho})C)e(t)\\ &+Ff(t))^T\times j\omega_0 Ee(t)\end{aligned} \tag{40}$$

where $\omega_0=\frac{\omega_1+\omega_2}{2}$.

Based (32), (33), and (40), and $\mathfrak{J}_-<0$ leads to (17).

Remark 1: (15) and (17) in Theorem 1 are not linear matrix inequalities, making the solving process difficult. Theorem 2 is provided, which reorganizes the existence conditions for the H_-/L_∞ fault detection observer into the form of linear matrix inequalities [15].

Theorem 2: For the T-S fuzzy system shown in (9), given scalars $\beta>0,\gamma_1,\gamma_2>0,\alpha_1,\alpha_2$, and the matrix $V_f\in\mathbb{R}^{n_x\times n_y}$, for any $i=1,...,64$, if there exist scalars $\alpha>0$ and matrices $P_{di}=P_{di}^T>0$, $P_{fi}=P_{fi}^T>0\in\mathbb{R}^{n_x\times n_x}$, and $E=E^T\in\mathbb{R}^{n_x\times n_x}$, $M\in\mathbb{R}^{n_x\times n_x}$, and $W_i\in\mathbb{R}^{n_x\times n_y}$ such that the following inequalities hold:

$$\begin{bmatrix}\Pi_1 & M & \Pi_2\\ \star & -\alpha I & \alpha_1 M^T\\ \star & \star & -\alpha_1 M--\alpha_1 M^T\end{bmatrix}<0 \tag{41}$$

$$\begin{bmatrix}C^T C-\gamma_1^2 P_{di} & C^T D_y\\ D_y^T C & -\gamma_2^2+D_y^T D_y\end{bmatrix}<0 \tag{42}$$

$$\begin{bmatrix}\Omega_1 & \Omega_2 & \Omega_3\\ \star & \Omega_4 & -V_f^T M+\alpha_2 F^T M^T\\ \star & \star & -E-\alpha_2 M-\alpha_2 M^T\end{bmatrix}<0 \tag{43}$$

such that the observer (11) satisfies the conditions specified in (18) and (19). Then, the observer (11) is an H_-/L_∞ observer, and the observer gain is $L_i=M^{-1}W_i$, where

$$\begin{cases}\Pi_1=\alpha P_{di}+(MA_i-W_iC)^T+(MA_i-W_iC)\\ \Pi_2=-M+P_{di}+\alpha_1(MA_i-W_i)^T\\ \Omega_1=-\omega_1\omega_2 E-C^T C+(MA_i-W_iC)+(MA_i-W_iC)^T\\ \Omega_2=MF+(MA_i-W_iC)^T V_f\\ \Omega_3=-M+P_{fi}-j\omega_0 E+\alpha_2(MA_i-W_iC)^T\\ \Omega_4=\beta^2 I+V_f^T MF+F^T M^T V_f\end{cases}$$

Proof: (15) can be equivalently written as:

$$\mathcal{M}_1+\mathcal{P}_1\mathcal{A}_1+\mathcal{A}_1^T\mathcal{P}_1^T<0 \tag{44}$$

where

$$\mathcal{M}_1=\begin{bmatrix}\alpha P_d(\hat{\rho}) & 0\\ 0 & -\alpha I\end{bmatrix}\mathcal{P}_1=\begin{bmatrix}P_d(\hat{\rho})\\ 0\end{bmatrix}\mathcal{A}_1=\begin{bmatrix}\mathbb{A}_d & I\end{bmatrix} \tag{45}$$

Inequality (44) is equivalently written as:

$$\begin{bmatrix}\mathcal{M}_1+\Psi_1\mathcal{A}_1+\mathcal{A}_1^T\Psi_1^T & -\Psi_1+\mathcal{P}_1+\mathcal{A}_1^T G_d^T\\ \star & -G_d-G_d^T\end{bmatrix}<0 \tag{46}$$

where

$$\Psi_1 = \begin{bmatrix} M \\ 0 \end{bmatrix}, W_i = ML_i, G_d = \alpha_1 M \tag{47}$$

By substituting (47) into (46), we obtain equation (41). Similarly, equation (17) is equivalent to

$$\mathcal{M}_2 + \mathcal{A}_2^T \mathcal{P}_2^T + \mathcal{P}_2 \mathcal{A}_2 - \mathcal{A}_2^T S \mathcal{A}_2 < 0 \tag{48}$$

where

$$\mathcal{M}_2 = \begin{bmatrix} -\omega_1\omega_2 S - C^T C & 0 \\ 0 & \beta^2 I \end{bmatrix}, \mathcal{P}_2 = \begin{bmatrix} P_f(\hat{\rho}) - j\omega_0 E \\ 0 \end{bmatrix},$$
$$\mathcal{A}_2 = \begin{bmatrix} (A(\hat{\rho}) - L(\hat{\rho})C) & F \end{bmatrix} \tag{49}$$

Equation (48) is also equivalent to:

$$\begin{bmatrix} \mathcal{M}_2 + \Psi_2 \mathcal{A}_2 + \mathcal{A}_2^T \Psi_2^T & -\Psi_2 + \mathcal{P}_2 + \mathcal{A}_2^T G_f^T \\ \star & -E - G_f - G_f^T \end{bmatrix} < 0 \tag{50}$$

where

$$\Psi_2 = \begin{bmatrix} M \\ V_f^T M \end{bmatrix}, G_f = \alpha_2 M, W_i = ML_i \tag{51}$$

By substituting (51) into (50), we can obtain (43).

III-B. Residual evaluation and threshold computation

After the residual is generated, selecting the correct residual evaluation function and its corresponding threshold is crucial for the result of fault detection. In this paper, the L_2 norm of the residual at time t, $\|r(t)\|$ is chosen as the residual evaluation function, that is

$$\mathfrak{Y}(t) = \|r(t)\| \tag{52}$$

Select the decision threshold as

$$\mathfrak{Y}_{\text{th}}(t) = \sup_{f(t)=0} \|r(t)\| \tag{53}$$

Compare the residual evaluation function with the threshold to determine whether a fault has occurred, i.e.

$$\begin{cases} \mathfrak{Y}(t) > \mathfrak{Y}_{\text{th}}(t) \Rightarrow \text{The fault occurs,} \\ \mathfrak{Y}(t) \leqslant \mathfrak{Y}_{\text{th}}(t) \Rightarrow \text{No fault occurs.} \end{cases} \tag{54}$$

IV. SIMULATION RESULT

This section presents simulation experiments to validate the feasibility of the proposed method. The system parameters are provided in Table II.

TABLE II: SIMULATION PARAMETERS

Symbol	Value	Symbol	Value
ϱ	0.65	q_r	0.6
$\mathfrak{K}_0$	0.05	κ	0.2
$C_{s_{\text{in}}}$	200(mg/l)	$\vartheta_{\max}$	0.15 (h^{-1})
$C_{0_{\text{in}}}$	0.5(mg/l)	$\mathfrak{K}_s$	100(mg/l)
$C_{0_{\max}}$	10(mg/l)	$\mathfrak{K}_a$	2(mg/l)

We first compute the 64 fuzzy sub-systems of system (8), the matrices D_x and D_y are given by $D_x = [0\ 0\ 0\ 1]^T$ and $D_y = [0\ 0\ 0\ 0]$, respectively. A subset of the computed results for the 64 fuzzy sub-systems is given below as examples.

$$A_1 = \begin{bmatrix} -5 & 0 & 300 & 0 \\ 0 & -9.5 & 0 & 400 \\ 0 & 0 & 0 & -7.69 \\ -400 & 300 & 0 & -3.85 \end{bmatrix}, \dots,$$
$$A_{64} = \begin{bmatrix} -120 & 0 & 60 & 0 \\ 0 & -0.4 & 0 & 200 \\ 0 & 0 & 0 & -0.02 \\ -200 & 60 & 0 & -0.01 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -240 & 0 \\ 160 & 0 \\ 8 & 0 \\ -14.7 & 0.5 \end{bmatrix}, \dots, B_{64} = \begin{bmatrix} 24 & 0 \\ -64 & 0 \\ 184 & 0 \\ -0.14 & 9.6 \end{bmatrix}$$

The parameters are set as follows: $\gamma_1 = 2.5$, $\gamma_2 = 0.9$, $\omega_1 = 0.6$, $\omega_2 = 3$, $\alpha_1 = 0.9$, $\alpha_2 = 0.33$, and $V_f = -0.6B_1$. The L_∞ performance index is chosen as $\alpha = 2.5$, and the H_- performance index is selected as $\beta = 1.3$. Then, the LMI conditions in Theorem 2 are solved using the YALMIP toolbox in MATLAB, and the observer gain L for each fuzzy sub-system can be obtained.

$$L_1 = \begin{bmatrix} -8165.1 \\ 3800.8 \\ -439.4 \\ 4550.5 \end{bmatrix}, \dots, L_{64} = \begin{bmatrix} -8138.8 \\ 3524 \\ -454.2 \\ 4544.2 \end{bmatrix}$$

In this paper, we focus on two common types of actuator fault signals: time-invariant fault signals and time-varying fault signals. The specific forms are given as follows.

Case 1: Time-invariant actuator fault signal: The fault occurs on the $t = 30$ days of operation, and the fault signal amplitude remains constant thereafter.

$$f_a(t) = \begin{cases} 0, t < 30 \\ 0.4, 30 \le t \end{cases} \tag{55}$$

Case 2: Time-varying actuator fault: The fault occurs after $t = 35$ days, and the fault signal amplitude varies over time.

$$f_b(t) = \begin{cases} 0, t < 35 \\ 0.36(t-35), 35 \le t \end{cases} \tag{56}$$

To verify the effectiveness of the proposed method, the proposed H_-/L_∞ method is compared with the conventional L_∞ observer design method.

As shown in Figure 1, Case 1 presents an abrupt fault that remains constant over time. The dashed line represents the fault diagnosis result for the wastewater treatment system using the proposed H_-/L_∞ method, while the solid line represents the fault diagnosis result using the L_∞ method. The proposed method clearly outperforms the conventional L_∞ method, as it can fully detect the occurrence of faults. Figure 2 shows a time-varying fault. It is evident from the figure that the

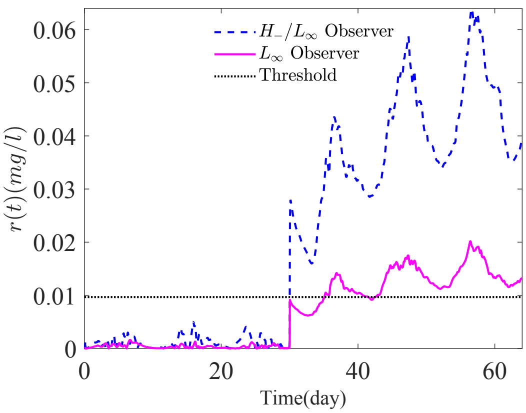

Fig. 1: fault detection result for $f_a(t)$

method discussed in this paper, by incorporating the H_- performance index, can detect the fault earlier.

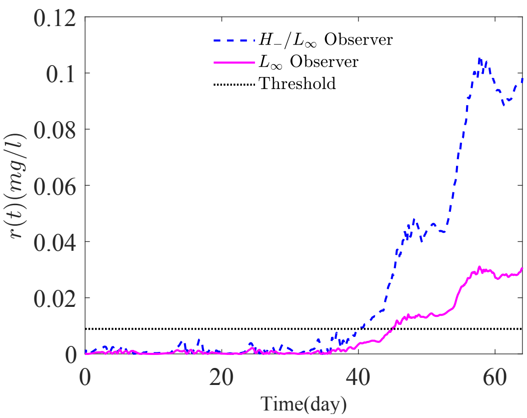

Fig. 2: fault detection result for $f_b(t)$

V. CONCLUSIONS

This paper proposes a fault detection method based on the H_-/L_∞ observer for the nonlinear wastewater treatment process, where unmeasurable premise variables are treated as disturbances during the observer design. The occurrence of actuator faults is assumed to be within a finite frequency range. The H_- performance index is used to assess fault sensitivity, while the L_∞ performance index analyzes the influence of disturbances on the observer. Additionally, the design conditions for the H_-/L_∞ observer are formulated as linear matrix inequalities to simplify the solution process, and the effectiveness of the proposed method is verified through simulation results. In future research, it is important to further reduce the conservatism in observer design. This will help improve the performance of fault detection and enhance its applicability in practical engineering systems.

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