Optimal regulator for linear stochastic systems with Markovian-switching coefficients and state-delay

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Abstract— We consider an optimal control problem for linear stochastic systems with *Markovian-switching* coefficients, *state-delay*, additive and multiplicative noise. By developing a generalized quadratic-linear cost functional that includes Markovian-switching coefficients, and employing the method of completing the squares, we succeed in solving the aforementioned problem in an explicit closed-form solution through a system of coupled Riccati and partial differential equations. The optimal control law is of an *affine* feedback form with respect to the system state, the delayed state, and the integral of the previous system state values.

I. INTRODUCTION

Let $(\Omega, \mathscr{F}, (\mathscr{F}(t), t \geq 0), \mathbb{P})$ be a complete filtered probability space on which the one-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. Further, let $(\xi(t), t \geq 0)$ be a right continuous Markov chain defined on this probability space, independent of W, taking values in the finite space $\mathbb{K} := \{1, ..., \kappa\}$, and with its generator $\Lambda = (\gamma_{ij})_{\kappa \times \kappa}$ defined as:

$$\mathbb{P}\{\xi(t+\delta) = j | \xi(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j \\ 1 + \gamma_{ij}\delta + o(\delta) & \text{if } i = j \end{cases}$$

where γ_{ij} is the transition rate from state i to state j if $i \neq j$, and

$$\gamma_{ij} = -\sum_{i \neq j} \gamma_{ij}.$$

We assume that $\mathscr{F}(t)$ is the augmentation of $\sigma(\xi(s), W(s) : 0 \leq s \leq t)$ by all the null sets of $\mathscr{F}$. Consider the following *linear* stochastic control system with *Markovian-switching* coefficients (for $t \in [0,T]$):

$$\begin{cases} dx(t) = [A^*(t,\xi(t))x(t) + B^*(t,\xi(t))u(t)]dt \\ \quad + [\underline{A}^*(t,\xi(t))x(t) + \underline{B}^*(t,\xi(t))u(t)]dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad \xi(0) = \xi_0 \in \mathbb{K}, \quad \text{are given,} \end{cases} \quad (1)$$

The coefficients of (1) are assumed as: $A^*(\cdot,i)$, $\underline{A}^*(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n \times n})$, and $B^*(\cdot,i)$, $\underline{B}^*(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n \times m})$, for $i = 1, ..., \kappa$, where $L^\infty(0,T;E)$ denotes the set of E-valued uniformly bounded functions (with E being an Euclidean space). The m-dimensional control process u is such that equation (1) has a unique strong solution for the n-dimensional system state x on the interval $[0,T]$. We associate with (1) the following *quadratic* cost functional:

$$I(u(\cdot)) := \mathbb{E}\left\{\int_0^T [x'(t)Q^*(t,\xi(t))x(t) + u'(t)R^*(t,\xi(t))u(t)]dt \right.$$
$$\left. + x'(T)H^*(\xi(T))x'(T) \big| \xi(0) = i \right\}, \quad (2)$$

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where $Q^*(\cdot,i) \in L^\infty(0,T;\mathbb{S}^n)$, $R^*(\cdot,i) \in L^\infty(0,T;\mathbb{S}^m)$, $H(i) \in \mathbb{S}^{n \times n}$, $i = 1, ..., \kappa$, with $\mathbb{S}^n$ and $\mathbb{S}^n$ denoting the sets of real valued $n \times n$ and $m \times m$ symmetric matrices, respectively. The prime symbol $'$ denotes the transpose of a vector. The optimal stochastic control problem:

$$\begin{cases} \min_{u(\cdot) \in \mathscr{A}^*} I(\cdot) \\ s.t. \quad (1) \end{cases} \quad (3)$$

for some suitable admissible set of controls $\mathscr{A}^*$, is the well-known stochastic *linear quadratic* (LQ) optimal control problem with Markovian switching. One of the first solutions to this problem was given in [13], whereas [14] considers its infinite horizon version, and [7] is a textbook account of the basic theory of this problem. An important characteristic of (1) is that it admits a unique closed-form solution in a linear state-feedback form, the gain of which is determined by a system of *coupled* Riccati differential equations (the solvability of which is typically assumed). This problem continues to be studied in various different stochastic settings, see, e.g., the recent papers [22], [9], [17] [18], where in particular the open-loop solutions and the random coefficients are considered.

In this paper, we *generalise* the problem (3) by introducing the *state-delay* into the system dynamics (1) and the cost functional (2) as follows. Instead of the system (1), we consider the following linear stochastic control system with Markovian regime switching coefficients and state-delay (for $t \in [0,T]$):

$$\begin{cases} dx(t) = [A(t,\xi(t))x(t) + B(t,\xi(t))x(t-h) \\ \quad + C(t,\xi(t))u(t) + D(t,\xi(t))]dt + [\bar{A}(t,\xi(t))x(t) \\ \quad + \bar{B}(t,\xi(t))x(t-h) + \bar{C}(t,\xi(t))u(t) + \bar{D}(t,\xi(t))]dW(t) \\ x(s) = \eta(s), \quad \xi(s) = i \in \mathbb{K}, \quad s \in [-h,0], \text{ are given,} \end{cases} \quad (4)$$

where $0 \leq h \in \mathbb{R}$ is the system delay, and the n-dimensional initial value η is assumed to be a continuous function. For the system coefficients we assume the following: $A(\cdot,i)$, $B(\cdot,i)$, $\bar{A}(\cdot,i)$, $\bar{B}(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n \times n})$; $C(\cdot,i)$, $\bar{C}(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n \times m})$; $D(\cdot,i)$, $\bar{D}(\cdot,i) \in L^\infty(0,T;\mathbb{R}^n)$, for $i = 1, ..., \kappa$. It is clear that (4), compared to (1), is more general, as it allows for Markovian regime switching coefficients, additive noise is included, i.e. $\bar{D}$ is not necessarily zero, and a *disturbance D* is permitted in the system. Instead of the cost functional (2), consider further the following criterion:

$$J(u(\cdot)) := \mathbb{E}\left\{\int_0^T [x'(t)Q_{11}(t,\xi(t))x(t) + x'(t)Q_{12}(t,\xi(t))x(t-h)\right.$$

$$+x'(t-h)Q_{22}(t,\xi(t))x(t-h)+x'(t-h)Q_{21}(t,\xi(t))x(t)$$
$$+u'(t)R(t,\xi(t))u(t)+u'(t)L_1(t,\xi(t))x(t)$$
$$+u'(t)L_2(t,\xi(t))x(t-h)+M_1'(t,\xi(t))x(t)$$
$$+M_2'(t,\xi(t))x(t-h)+S'(t,\xi(t))u(t)\big]dt$$
$$+x'(T)H(\xi(T))x(T)+K'(\xi(T))x(T)\big|\xi(0)=i\Big\}. \quad (5)$$

Here $Q_{11}(\cdot,i),Q_{22}(\cdot,i)\in L^\infty(0,T;\mathbb{S}^n)$, $Q_{12}(\cdot,i),Q_{21}'(\cdot,i)\in L^\infty(0,T;\mathbb{R}^{n\times n})$, $R(\cdot,i)\in L^\infty(0,T;\mathbb{S}^m)$, $L_1(\cdot,i)$, $L_2(\cdot,i)\in L^\infty(0,T;\mathbb{R}^{m\times n})$, $M_1(\cdot,i)$, $M_2(\cdot,i)\in L^\infty(0,T;\mathbb{R}^n)$, $S(\cdot,i)\in L^\infty(0,T;\mathbb{R}^m)$, $H(i)\in\mathbb{S}^n$, $K(i)\in\mathbb{R}^n$, $Q_{12}(t)=Q_{21}'(t)$ for $t\in[0,T]$, $i=1,...,\kappa$. Thus, the cost functional (5) is more general than (2) in the following: the coefficients can be Markovian regime switching coefficients; it includes a penalty on the delayed state $x(t-h)$, the cross products (between the state, delayed state, and control) and liner terms on the state and control. The *optimal regulator problem* to be considered is:

$$\begin{cases} \min_{u(\cdot)\in\mathscr{A}} J(u(\cdot)), \\ \text{s.t. } (4). \end{cases} \quad (6)$$

Here $\mathscr{A}:=L_\mathscr{F}^2(0,T;\mathbb{R}^m)$, i.e. the space of all real-valued m-dimensional square-integrable adapted processes, and this in particular ensures that if $u(\cdot)\in\mathscr{A}$, then there exist of a unique strong solution to (4) (see, e. g., [16]). The problem (6), in addition to being a state-delay generalisation to (3), it is also a Markovain switching generalisation of the stochastic LQ control problem with state-delay, which has been studied extensively in recent years, see, for example, [1], [2], [5], [4], [6], [10], [11], [12], [15], [19]), [20], [21]). The motivation for considering this problem is as follows: it is an important optimal control problem for which we can find an explicit closed-form solution, which turns out to be of an *affine* feedback form with respect to the system state, the delayed state, and the integral of past system state values. It also has a potential for applications, in particular in problems of optimal investment (see, e.g., [3], [8], [23], [9], for some possible applications). Our approach to finding the solution is a generalisation of the approach in [1] and [2], and it is given in §II below. In what follows, we omit the argument t whenever convenient for notation simplicity.

II. SOLUTION TO THE OPTIMAL REGULATOR PROBLEM

In order to state the solution to optimal regulator problem (6), we introduce two sets of (coupled) linear and Riccati ordinary differential equations which are coupled with a system of (coupled) partial differential equations. Firstly, on the intervals $t\in[T-h,T]$, $\theta\in[t,T]$, $s\in[t,T]$, $i=1,..,\kappa$, consider the equations:

$$\begin{cases} \dot{P}(t,i)+A'(t,i)P(t,i)+P(t,i)A(t,i) \\ +\bar{A}'(t,i)P(t,i)\bar{A}(t,i)-G_2'(t,i)\bar{R}^{-1}(t,i) \\ \times G_2(t,i)+Q_{11}(t,i)+\sum_{j=1}^\kappa \gamma_{ij}P(t,j)=0, \\ \bar{R}(t,i):=R(t,i)+\bar{C}'(t,i)P(t,i)\bar{C}(t,i)>0, \\ P(T,i)=H(i), \end{cases} \quad (7)$$

$$\begin{cases} \dot{f}(t,i)+q'(t,i)D(t,i)+\bar{D}'(t,i)P(t,i)\bar{D}(t,i) \\ -G_1'(t,i)\bar{R}^{-1}(t,i)G_1(t,i)=0, \\ f(T,i)=0, \end{cases} \quad (8)$$

$$\begin{cases} \dot{q}'(t,i)+q'(t,i)A(t,i)+2D'(t,i)P(t,i)+2\bar{D}'(t,i)P(t,i) \\ \times\bar{A}(t,i)-2G_1'(t,i)\bar{R}^{-1}(t,i)G_2(t,i)+M_1'(t,i) \\ +\sum_{j=1}^\kappa \gamma_{ij}q'(t,j)=0, \\ q'(T,i)=K'(i), \end{cases} \quad (9)$$

$$\begin{cases} \dfrac{\partial N_1'(t,\theta,i)}{\partial t}+2D'(t,i)N_2(t,\theta,i)-2G_1'(t,i) \\ \times\bar{R}^{-1}(t,i)G_4(t,\theta,i)+\sum_{j=1}^\kappa \gamma_{ij}N_1'(t,\theta,j)=0, \\ q'(t,i)B(t,i)+2\bar{D}'(t,i)P(t,i)\bar{B}(t,i)-2G_1'(t,i) \\ \times\bar{R}^{-1}(t,i)G_3(t,i)-N_1'(t,t,i)+M_2'(t,i)=0, \end{cases} \quad (10)$$

$$\begin{cases} \dfrac{2\partial N_2(t,\theta,i)}{\partial t}+2A'(t,i)N_2(t,\theta,i)-2G_2'(t,i) \\ \bar{R}^{-1}(t,i)G_4(t,\theta,i)+\sum_{j=1}^\kappa \gamma_{ij}N_2(t,\theta,j)=0, \\ 2P(t,i)B(t,i)+2\bar{A}'(t,i)P(t,i)\bar{B}(t,i)-2N_2(t,t,i) \\ -2G_2'(t,i)\bar{R}^{-1}(t,i)G_3(t,i)+2Q_{12}(t,i)=0, \end{cases} \quad (11)$$

$$\begin{cases} \dfrac{\partial N_3(t,\theta,i)}{\partial t}+\sum_{j=1}^\kappa \gamma_{ij}N_3(t,\theta,j)=0, \\ \bar{B}'(t,i)P(t,i)\bar{B}(t,i)-N_3(t,t,i)-G_3'(t,i)\bar{R}^{-1}(t,i) \\ \times G_3(t,i)+Q_{22}(t,i)=0, \end{cases} \quad (12)$$

$$\begin{cases} \dfrac{\partial N_4(t,s,\theta,i)}{\partial t}-G_4'(t,\theta,i)\bar{R}^{-1}(t,i)G_4(t,s,i) \\ +\sum_{j=1}^\kappa \gamma_{ij}N_4(t,s,\theta,j)=0, \\ B'(t,i)N_2(t,\theta,i)-N_4(t,\theta,t,i)-G_3'(t,i)\bar{R}^{-1}(t,i) \\ \times G_4(t,\theta,i)=0, \\ N_2'(t,\theta,i)B(t,i)-N_4(t,t,\theta,i)-G_4'(t,\theta,i) \\ \times\bar{R}^{-1}(t,i)G_3(t,i)=0, \end{cases} \quad (13)$$

where

$$G_1(t,i):=0.5C'(t,i)q(t,i)+\bar{C}'(t,i)P(t,i)$$
$$\times\bar{D}(t,i)+0.5S(t,i),$$
$$G_2(t,i):=C'(t,i)P(t,i)+\bar{C}'(t,i)P(t,i)\bar{A}(t,i)$$
$$+0.5L_1(t,i),$$
$$G_3(t,i):=\bar{C}'(t,i)P(t,i)\bar{B}(t,i)+0.5L_2(t,i),$$
$$G_4(t,\theta,i):=C'(t,i)N_2(t,\theta,i).$$

Secondly, on the intervals $t\in[0,T-h]$, $\theta\in[t,t+h]$, $s\in[t,t+h]$, $i=1,...,\kappa$, consider the equations:

$$\begin{cases} \dot{\bar{P}}(t,i)+Q_{11}(t,i)-\bar{G}_2'(t,i)\underline{R}^{-1}(t,i)\bar{G}_2(t,i) \\ +\bar{N}_2(t,t+h,i)+\bar{N}_2'(t,t+h,i)+\bar{A}'(t,i) \\ \times\bar{P}(t,i)\bar{A}(t,i)+A'(t,i))\bar{P}(t,i)+\bar{P}(t,i) \\ \times A(t,i)+\bar{N}_3(t,t+h,i)+\sum_{j=1}^\kappa \gamma_{ij}\bar{P}(t,j)=0, \\ \underline{R}(t,i):=R(t,i)+\bar{C}'(t,i)\bar{P}(t,i)\bar{C}(t,i)>0, \\ \bar{P}(T-h,i)=P(T-h,i), \end{cases} \quad (14)$$

$$\begin{cases} \dot{\bar{f}}(t,i)+\bar{q}'(t,i)D(t,i)+\bar{D}'(t,i)\bar{P}(t,i)\bar{D}(t,i)-\bar{G}_1'(t,i) \\ \times\underline{R}^{-1}(t,i)\bar{G}_1(t,i)=0 \\ \bar{f}(T-h,i)=f(T-h,i), \end{cases} \quad (15)$$

$$\begin{cases} \ddot{q}'(t,i)+\bar{q}'(t,i)A(t,i)+2D'(t,i)\bar{P}(t,i) \\ \quad +2\bar{D}'(t,i)\bar{P}(t,i)\bar{A}(t,i)+\bar{N}'_1(t,t+h,i)-2\bar{G}'_1(t,i) \\ \quad \times \underline{R}^{-1}(t,i)\bar{G}_2(t,i)+M'_1(t,i)+\sum_{j=1}^{\kappa}\gamma_{ij}\bar{q}'(t,j)=0, \\ \bar{q}'(T-h,i)=q'(T-h,i), \end{cases}$$
(16)

$$\begin{cases} \dfrac{\partial \bar{N}'_1(t,\theta,i)}{\partial t}+2D'(t,i)\bar{N}_2(t,\theta,i)-2\bar{G}'_1(t,i) \\ \quad \times \underline{R}^{-1}(t,i)\bar{G}_4(t,\theta,i)+\sum_{j=1}^{\kappa}\gamma_{ij}\bar{N}'_1(t,\theta,j)=0, \\ \bar{q}'(t,i)B(t,i)+2\bar{D}'(t,i)\bar{P}(t,i)\bar{B}(t,i)-\bar{N}'_1(t,t,i) \\ \quad -2\bar{G}'_1(t,i)\underline{R}^{-1}(t,i)\bar{G}_3(t,i)+M'_2(t,i)=0, \\ \bar{N}_1(T-h,z,i)=N_1(T-h,z,i) \quad \text{for} \quad z\in[T-h,T], \end{cases}$$
(17)

$$\begin{cases} 2\dfrac{\partial \bar{N}'_2(t,\theta,i)}{\partial t}+2A'(t,i)\bar{N}_2(t,\theta,i)+\bar{N}_4(t,\theta,t+h,i) \\ \quad +\bar{N}'_4(t,t+h,\theta,i)+\sum_{j=1}^{\kappa}\gamma_{ij}\bar{N}'_2(t,\theta,j)=0, \\ 2B'(t,i)\bar{P}(t,i)+2\bar{B}'(t,i)\bar{P}(t,i)\bar{A}(t,i)-2\bar{N}'_2(t,t,i) \\ \quad -2\bar{G}'_3(t,i)\underline{R}^{-1}(t,i)\bar{G}_2(t,i)+2Q_{12}(t,i)=0, \\ \bar{N}_2(T-h,z,i)=N_2(T-h,z,i) \quad \text{for} \quad z\in[T-h,T], \end{cases}$$
(18)

$$\begin{cases} \dfrac{\partial \bar{N}_3(t,\theta,i)}{\partial t}+\sum_{j=1}^{\kappa}\gamma_{ij}\bar{N}'_3(t,\theta,j)=0, \\ \bar{B}'(t,i)\bar{P}(t,i)\bar{B}(t,i)-\bar{N}_3(t,t,i)-\bar{G}'_3(t,i) \\ \quad \times \underline{R}^{-1}(t,i)\bar{G}_3(t,i)+Q_{22}(t,i)=0, \\ \bar{N}_3(T-h,z,i)=N_3(T-h,z,i) \quad \text{for} \quad z\in[T-h,T], \end{cases}$$
(19)

$$\begin{cases} \dfrac{\partial \bar{N}_4(t,s,\theta,i)}{\partial t}-\bar{G}'_4(t,\theta,i)\underline{R}^{-1}(t,i)\bar{G}_4(t,s,i) \\ \quad +\sum_{j=1}^{\kappa}\gamma_{ij}\bar{N}'_4(t,s,\theta,j)=0, \\ B'(t,i)\bar{N}_2(t,\theta,i)-\bar{N}_4(t,\theta,t,i)-\bar{G}'_3(t,i)\underline{R}^{-1}(t,i) \\ \quad \times \bar{G}_4(t,\theta,i)=0, \\ \bar{N}'_2(t,\theta,i)B(t,i)-\bar{N}_4(t,t,\theta,i)-\bar{G}'_4(t,\theta,i) \\ \quad \times \underline{R}^{-1}(t,i)\bar{G}_3(t,i)=0, \\ \bar{N}_4(T-h,\ell,z)=N_4(T-h,\ell,z) \quad \text{for} \quad \ell\in[T-h,T] \\ \text{and} \quad z\in[T-h,T], \end{cases}$$
(20)

where

$$\begin{aligned} \bar{G}_1(t,i) &:= 0.5C'(t,i)\bar{q}(t,i)+\bar{C}'(t,i)\bar{P}(t,i)\bar{D}(t,i) \\ &\quad +0.5S(t,i), \\ \bar{G}_2(t,i) &:= C'(t,i)\bar{P}(t,i)+\bar{C}'(t,i)\bar{P}(t,i)\bar{A}(t,i) \\ &\quad +0.5L_1(t,i), \\ \bar{G}_3(t,i) &:= \bar{C}'(t,i)\bar{P}(t,i)\bar{B}(t,i)+0.5L_2(t,i), \\ \bar{G}_4(t,\theta,i) &:= C'(t,i)\bar{N}_2(t,\theta,i). \end{aligned}$$

The system of coupled equations (7)-(20) is more general than the ones that appear in [1], due to the additional coupling of equations (that comes from Markovian switching). Their unique solvability is our standing assumption below.

Assumption 1. *The system of coupled equations (7)-(20) has a unique solution.*

Theorem 2.1: There exists a unique solution u^* to the optimal regulator problem (6). If $T-h\leq 0$, then the solution is given by:

$$\begin{aligned} u^*(t) &= -\bar{R}^{-1}(t,\xi(t))\left[G_1(t,\xi(t))+G_2(t,\xi(t))x(t)\right. \\ &\quad \left.+\,G_3(t,\xi(t))\right]x(t-h)-\bar{R}^{-1}(t,\xi(t))\int_t^T G_4(t,\theta,\xi(t)) \\ &\quad \times x(\theta-h)d\theta, \quad t\in[0,T]. \end{aligned}$$

If $T-h>0$, then the solution is given by:

$$\begin{aligned} u^*(t) &= -\underline{R}^{-1}(t,\xi(t))\left[\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)\right. \\ &\quad +\, \bar{G}_3(t,\xi(t))x(t-h)\left]-\underline{R}^{-1}(t,\xi(t))\right. \\ &\quad \times \int_t^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta, \quad t\in[0,T-h], \\ u^*(t) &= -\bar{R}^{-1}(t,\xi(t))\left[G_1(t,\xi(t))+G_2(t,\xi(t))x(t)\right. \\ &\quad +\, G_3(t,\xi(t))x(t-h)\left]-\bar{R}^{-1}(t,\xi(t))\right. \\ &\quad \times \int_t^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta, \quad t\in[T-h,T]. \end{aligned}$$

Proof. We only consider the case of $T-h>0$, as the case of $T-h\leq 0$ is very similar and simpler. Thus, let $T-h>0$. We split (5) in two parts as $J(u(\cdot))=\mathbb{E}[J_1(u(\cdot))]+\mathbb{E}[J_2(u(\cdot))]$ where:

$$\begin{aligned} J_1(u(\cdot)):=\mathbb{E}\Bigg\{ &\int_0^{T-h}\left[x'(t)Q_{11}(t,\xi(t))x(t)+x'(t-h)\right. \\ &\times Q_{22}(t,\xi(t))x(t-h)+x'(t)Q_{12}(t,\xi(t))x(t-h) \\ &+x'(t-h)Q_{21}(t,\xi(t))x(t)+u'(t)R(t,\xi(t))u(t)+u'(t) \\ &\times L_1(t,\xi(t))x(t)+u'(t)L_2(t,\xi(t))x(t-h)+M'_1(t,\xi(t)) \\ &\times x(t)+M'_2(t,\xi(t))x(t-h)+S'(t,\xi(t))u(t)\Big]dt\bigg|\xi(0)=i\Bigg\}, \end{aligned}$$

$$\begin{aligned} J_2(u(\cdot)):=\mathbb{E}\Bigg\{ &\int_{T-h}^T\left[x'(t)Q_{11}(t,\xi(t))x(t)+x'(t-h)\right. \\ &\times Q_{22}(t,\xi(t))x(t-h)+x'(t)Q_{12}(t,\xi(t))x(t-h) \\ &+x'(t-h)Q_{21}(t,\xi(t))x(t)+u'(t)R(t,\xi(t))u(t)+u'(t) \\ &\times L_1(t,\xi(t))x(t)+u'(t)L_2(t,\xi(t))x(t-h)+M'_1(t,\xi(t)) \\ &\times x(t)+M'_2(t,\xi(t))x(t-h)+S'(t,\xi(t))u(t)\Big]dt \\ &+x'(T)Hx(T)+K'x(T)\bigg|\mathscr{F}(T-h)\Bigg\}. \end{aligned}$$

For $t\in[0,T-h]$, we define the process $v_1(t,x(t),\xi(t))$ as:

$$\begin{aligned} v_1(t,x(t),\xi(t)) &= \bar{f}(t,\xi(t))+\bar{q}'(t,\xi(t))x(t)+x'(t) \\ &\quad \times \bar{P}(t,\xi(t))x(t)+\int_t^{t+h}\bar{N}'_1(t,\theta,\xi(t))x(\theta-h)d\theta \\ &\quad +x'(t)\int_t^{t+h}\bar{N}_2(t,\theta,\xi(t))x(\theta-h)d\theta \\ &\quad +\int_t^{t+h}x'(\theta-h)\bar{N}'_2(t,\theta,\xi(t))x(t)d\theta+\int_t^{t+h}x'(\theta-h) \\ &\quad \times \bar{N}_3(t,\theta,\xi(t))x(\theta-h)d\theta+\int_t^{t+h}\int_t^{t+h}x'(\theta-h) \\ &\quad \times \bar{N}_4(t,s,\theta,\xi(t))x(s-h)d\theta ds. \end{aligned}$$

By the generalised Itô's formula (see, e. g., in [7], [16]), we obtain:

$$\begin{aligned} \mathbb{E}[v_1(t,x(t),\xi(t))|\xi(0)=i] &= v_1(0,x(0),i) \\ +\mathbb{E}\Bigg\{ &\int_0^{T-h}\left[\dot{\bar{f}}(t,\xi(t))dt+\ddot{q}'(t,\xi(t))x(t)dt\right. \\ &+\bar{q}'(t,\xi(t))[A(t,\xi(t))x(t)+B(t,\xi(t))x(t-h) \\ &+C(t,\xi(t))u(t)+D(t,\xi(t))]dt+\bar{q}'(t,\xi(t)) \\ &\times [\bar{A}(t,\xi(t))x(t)+\bar{B}(t,\xi(t))x(t-h)+\bar{C}(t,\xi(t)) \\ &\times u(t)+\bar{D}(t,\xi(t))]dW+\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{q}'(t,j)x(t) \end{aligned}$$

$$+x'(t)\dot{P}(t,\xi(t))x(t)dt+[A(t,\xi(t))x(t)+B(t,\xi(t))$$
$$\times x(t-h)+C(t,\xi(t))u(t)+D(t,\xi(t))]'\bar{P}(t,\xi(t))$$
$$\times x(t)dt+[\bar{A}(t,\xi(t))x(t)+\bar{B}(t,\xi(t))x(t-h)$$
$$+\bar{C}(t,\xi(t))u(t)+\bar{D}(t,\xi(t))]'\bar{P}(t,\xi(t))x(t)$$
$$\times dW+[\bar{A}(t,\xi(t))x(t)+\bar{B}(t,\xi(t))x(t-h)$$
$$+\bar{C}(t,\xi(t))u(t)+\bar{D}(t,\xi(t))]'\bar{P}(t,\xi(t))[\bar{A}(t,\xi(t))x(t)$$
$$+\bar{B}(t,\xi(t))x(t-h)+\bar{C}(t,\xi(t))u(t)+\bar{D}(t,\xi(t))]dt$$
$$+x'(t)\bar{P}(t,\xi(t))[A(t,\xi(t))x(t)+B(t,\xi(t))x(t-h)$$
$$+C(t,\xi(t))u(t)+D(t,\xi(t))]dt+x'(t)\bar{P}(t,\xi(t))$$
$$\times[\bar{A}(t,\xi(t))x(t)+\bar{B}(t,\xi(t))x(t-h)+\bar{C}(t,\xi(t))u(t)$$
$$+\bar{D}(t,\xi(t))]dW+x'(t)\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{P}(t,j)x(t)$$

$$+\int_{t}^{t+h}\frac{\partial\bar{N}_1'(t,\theta,\xi(t))}{\partial t}x(\theta-h)d\theta dt$$
$$-\bar{N}_1'(t,t,\xi(t))x(t-h)dt+\bar{N}_1'(t,t+h,\xi(t))x(t)dt$$
$$+\int_{t}^{t+h}\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_1'(t,\theta,j)x(\theta-h)d\theta$$

$$+x'(t)\int_{t}^{t+h}\frac{\partial\bar{N}_2(t,\theta,\xi(t))}{\partial t}x(\theta-h)d\theta dt$$
$$-x'(t)\bar{N}_2(t,t,\xi(t))x(t-h)dt+x'(t)\bar{N}_2(t,t+h,\xi(t))x(t)dt$$
$$+[A(t,\xi(t))x(t)+B(t,\xi(t))x(t-h)+C(t,\xi(t))u(t)$$
$$+D(t,\xi(t))]'\int_{t}^{t+h}\bar{N}_2(t,\theta,\xi(t))x(\theta-h)d\theta dt$$
$$+[\bar{A}(t,\xi(t))x(t)+\bar{B}(t,\xi(t))x(t-h)+\bar{C}(t,\xi(t))$$
$$\times u(t)+\bar{D}(t,\xi(t))]'\int_{t}^{t+h}\bar{N}_2(t,\theta,\xi(t))x(\theta-h)d\theta dW$$
$$+x'(t)\int_{t}^{t+h}\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_2(t,\theta,j)x(\theta-h)d\theta$$

$$+\int_{t}^{t+h}x'(\theta-h)\frac{\partial\bar{N}_2'(t,\theta,\xi(t))}{\partial t}x(t)d\theta dt-x'(t-h)$$
$$\times\bar{N}_2'(t,t,\xi(t))x(t)dt+x'(t)\bar{N}_2'(t,t+h,\xi(t))x(t)dt$$
$$+\int_{t}^{t+h}x'(\theta-h)\bar{N}_2'(t,\theta,\xi(t))d\theta[A(t,\xi(t))x(t)$$
$$+B(t,\xi(t))x(t-h)+C(t,\xi(t))u(t)+D(t,\xi(t))]dt$$
$$+\int_{t}^{t+h}x'(\theta-h)\bar{N}_2'(t,\theta,\xi(t))[\bar{A}(t,\xi(t))x(t)$$
$$+\bar{B}(t,\xi(t))x(t-h)+\bar{C}(t,\xi(t))u(t)+\bar{D}(t,\xi(t))]dW$$
$$+\int_{t}^{t+h}x'(\theta-h)\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_2(t,\theta,j)x(t)d\theta$$

$$+\int_{t}^{t+h}x'(\theta-h)\frac{\partial\bar{N}_3(t,\theta,\xi(t))}{\partial t}x(\theta-h)d\theta dt$$
$$-x'(t-h)\bar{N}_3(t,t,\xi(t))x(t-h)+x'(t)\bar{N}_3(t,t+h,\xi(t))$$
$$\times x(t)dt+\int_{t}^{t+h}x'(\theta-h)\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_3(t,\theta,j)x(\theta-h)d\theta$$

$$+\int_{t}^{t+h}\int_{t}^{t+h}x'(\theta-h)\frac{\partial\bar{N}_4(t,s,\theta,\xi(t))}{\partial t}x(s-h)d\theta ds dt$$
$$-\int_{t}^{t+h}x'(\theta-h)\bar{N}_4(t,t,\theta,\xi(t))x(t-h)d\theta dt$$

$$+\int_{t}^{t+h}x'(\theta-h)\bar{N}_4(t,t+h,\theta,\xi(t))x(t)d\theta dt$$
$$-x'(t-h)\int_{t}^{t+h}\bar{N}_4(t,\theta,t,\xi(t))x(\theta-h)d\theta dt$$
$$+x'(t)\int_{t}^{t+h}\bar{N}_4(t,\theta,t+h,\xi(t))x(\theta-h)d\theta dt$$
$$+\int_{t}^{t+h}\int_{t}^{t+h}x'(\theta-h)\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_4(t,s,\theta,j)$$
$$\times x(s-h)d\theta ds\bigg]dt\bigg|\xi(0)=i\bigg\}.$$

The terms of $\mathbb{E}[J_1(u(\cdot))+v_1(T-h,x(T-h),\xi(T-h))]$ that depend explicitly on control u can now be written as:
$$\bigg\{u(t)+\underline{R}^{-1}(t,\xi(t))\bigg[\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)$$
$$+\bar{G}_3(t,\xi(t))x(t-h)+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg]\bigg\}'$$
$$\times\underline{R}(t,\xi(t))\bigg\{u(t)+\underline{R}^{-1}(t,\xi(t))\bigg[\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))$$
$$\times x(t)+\bar{G}_3(t,\xi(t))x(t-h)+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)$$
$$\times d\theta\bigg]\bigg\}-\bigg[\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)+\bar{G}_3(t,\xi(t))$$
$$\times x(t-h)+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg]'\underline{R}^{-1}(t,\xi(t))$$
$$\times\bigg[\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)+\bar{G}_3(t,\xi(t))x(t-h)$$
$$+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg].$$

We thus have:
$$\mathbb{E}[J_1(u(\cdot))+v_1(T-h,x(T-h),\xi(T-h))|\xi(0)=i]$$
$$=v_1(0,x(0),i)+\mathbb{E}\bigg\{\int_{0}^{T-h}\bigg\{\bigg[u(t)+\underline{R}^{-1}(t,\xi(t))$$
$$\times\bigg(\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)+\bar{G}_3(t,\xi(t))x(t-h)$$
$$+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg)\bigg]'\underline{R}(t,\xi(t))$$
$$\times\bigg[u(t)+\underline{R}^{-1}(t,\xi(t))\bigg(\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))$$
$$\times x(t)+\bar{G}_3(t,\xi(t))x(t-h)+\int_{t}^{t+h}\bar{G}_4(t,\theta,\xi(t))$$
$$\times x(\theta-h)d\theta\bigg)\bigg]+\bigg[\dot{\bar{f}}(t,\xi(t))+\bar{q}'(t,\xi(t))D(t,\xi(t)).$$
$$+\bar{D}'(t,\xi(t))\bar{P}(t,\xi(t))\bar{D}(t,\xi(t))-\bar{G}_1(t,\xi(t))$$
$$\times\underline{R}^{-1}(t,\xi(t))\bar{G}_1(t,\xi(t))\bigg]+\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{q}'(t,j)$$
$$+\ddot{q}'(t,\xi(t))+\bar{q}'(t,\xi(t))A(t,\xi(t))+2D'(t,\xi(t))$$
$$\times\bar{P}(t,\xi(t))+2\bar{D}'(t,\xi(t))\bar{P}(t,\xi(t))\bar{A}(t,\xi(t))$$
$$+\bar{N}_1'(t,t+h,\xi(t))-2\bar{G}_1'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))$$
$$\bar{G}_2(t,\xi(t))+M_1'(t,\xi(t))\bigg]x(t)+\bigg[\bar{q}'(t,\xi(t))B(t,\xi(t))$$
$$+2\bar{D}'(t,\xi(t))\bar{P}(t,\xi(t))\bar{B}(t,\xi(t))-\bar{N}_1'(t,t,\xi(t))$$
$$-2\bar{G}_1'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_3(t,\xi(t))+M_2'(t,\xi(t))\bigg]$$

$$\times x(t-h)+x'(t-h)\bigg[2\bar{B}'(t,\xi(t))\bar{P}(t,\xi(t))$$
$$+2\bar{B}'(t,\xi(t))\bar{P}(t,\xi(t))\bar{A}(t,\xi(t))-2\bar{N}_2'(t,t,\xi(t))$$
$$-2\bar{G}_3'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_2(t,\xi(t))+2Q_{12}(t,\xi(t))\bigg]$$
$$\times x(t)+x'(t)\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{P}(t,j)+\dot{\bar{P}}(t,\xi(t))$$
$$+Q_{11}(t,\xi(t))-\bar{G}_2'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_2(t,\xi(t))$$
$$+\bar{N}_2(t,t+h,\xi(t))+\bar{N}_2'(t,t+h,\xi(t))+\bar{A}'(t,\xi(t))$$
$$\times\bar{P}(t,\xi(t))\bar{A}(t,\xi(t))+A'(t,\xi(t))\bar{P}(t,\xi(t))$$
$$+\bar{P}(t,\xi(t))A(t,\xi(t))+\bar{N}_3(t,t+h,\xi(t))\bigg]x(t)$$
$$+x'(t-h)\bigg[\bar{B}'(t,\xi(t))\bar{P}(t,\xi(t))\bar{B}(t,\xi(t))$$
$$-\bar{N}_3(t,t,\xi(t))-\bar{G}_3'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_3(t,\xi(t))$$
$$+Q_{22}(t,\xi(t))\bigg]x(t-h)+\int_t^{t+h}\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_1'(t,\theta,j)$$
$$+\frac{\partial\bar{N}_1'(t,\theta,\xi(t))}{\partial t}+2D'\bar{N}_2(t,\theta,\xi(t))$$
$$-2\bar{G}_1'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_4(t,\theta,\xi(t))\bigg]x(\theta-h)d\theta$$
$$+\int_t^{t+h}x'(\theta-h)\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_2'(t,\theta,j)+2\frac{\partial\bar{N}_2'(t,\theta,\xi(t))}{\partial t}$$
$$+2A'(t,\xi(t))\bar{N}_2(t,\theta,\xi(t))+\bar{N}_4'(t,\theta,t+h,\xi(t))$$
$$+\bar{N}_4(t,t+h,\theta,\xi(t))\bigg]x(t)d\theta+\int_t^{t+h}\int_t^{t+h}x'(\theta-h)$$
$$\times\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_4'(t,s,\theta,j)+\frac{\partial\bar{N}_4(t,s,\theta,\xi(t))}{\partial t}-\bar{G}_4'(t,\theta,\xi(t))$$
$$\times\underline{R}^{-1}(t,\xi(t))\bar{G}_4(t,s,\xi(t))\bigg]x(s-h)dsd\theta+\int_t^{t+h}x'(\theta-h)$$
$$\times\bigg[\sum_{j=1}^{\kappa}\gamma_{\xi(t)j}\bar{N}_3'(t,\theta,j)+\frac{\partial\bar{N}_3(t,\theta,\xi(t))}{\partial t}\bigg]x(\theta-h)d\theta$$
$$+\int_t^{t+h}x'(t-h)\bigg[2\bar{B}'(t,\xi(t))\bar{N}_2(t,\theta,\xi(t))-\bar{N}_4(t,\theta,t,\xi(t))$$
$$-\bar{N}_4'(t,t,\theta,\xi(t))-2\bar{G}_3'(t,\xi(t))\underline{R}^{-1}(t,\xi(t))\bar{G}_4(t,\theta,\xi(t))\bigg]$$
$$\times x(\theta-h)d\theta\bigg\}dt\bigg\}=v_1(0,x(0),\xi(0))+\mathbb{E}\bigg\{\int_0^{T-h}\bigg[u(t)$$
$$+\underline{R}^{-1}(t,\xi(t))\Big(\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))x(t)$$
$$+\bar{G}_3(t,\xi(t))x(t-h)+\int_t^{t+h}\bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta\Big)\bigg]'$$
$$\times R(t,\xi(t))\bigg[u(t)+\underline{R}^{-1}(t,\xi(t))\Big(\bar{G}_1(t,\xi(t))+\bar{G}_2(t,\xi(t))$$
$$\times x(t)+\bar{G}_3(t,\xi(t))x(t-h)+\int_t^{t+h}\bar{G}_4(t,\theta,\xi(t))$$
$$\times x(\theta-h)d\theta\Big)\bigg]dt\bigg|\xi(0)=i\bigg\}. \tag{21}$$

For $t\in[T-h,T]$, we define the process $v_2(t,x(t),\xi(t))$ as:
$$v_2(t,x(t),\xi(t)):=f(t,\xi(t))+q'(t,\xi(t))x(t)+x'(t)$$
$$P(t,\xi(t))x(t)+\int_t^T N_1'(t,\theta,\xi(t))x(\theta-h)d\theta+x'(t)$$

$$\times\int_t^T N_2(t,\theta,\xi(t))x(\theta-h)d\theta+\int_t^T x'(\theta-h)N_2'(t,\theta,\xi(t))$$
$$\times x(t)d\theta+\int_t^T x'(\theta-h)N_3(t,\theta,\xi(t))x(\theta-h)d\theta$$
$$+\int_t^T\int_t^T x'(\theta-h)N_4(t,s,\theta,\xi(t))x(s-h)d\theta ds.$$

The differential of $v_2(t,x(t))$ can be obtained in a similar way to that of $v_1(t,x(t))$, by applying the generalised Itô's formula. The full derivation is omitted due to space restrictions. Further, the cost functional $J_2(u(\cdot))$ can be written as:

$$J_2(u(\cdot)):=\mathbb{E}\bigg\{\int_{T-h}^T[x'(t)(Q_{11}(t,\xi(t))x(t)+x'(t-h)$$
$$\times Q_{22}(t,\xi(t))x(t-h)+x'(t)Q_{12}(t,\xi(t))x(t-h)$$
$$+x'(t-h)Q_{21}(t,\xi(t))x(t)+u'(t)R(t,\xi(t))u(t)$$
$$+u'(t)L_1(t,\xi(t))x(t)+u'(t)L_2(t,\xi(t))x(t-h)$$
$$+M_1'(t,\xi(t))x(t)+M_2'(t,\xi(t))x(t-h)$$
$$+S'(t,\xi(t))u(t)\big]dt+v_2(T,x(T),\xi(T))\Big|\mathscr{F}(T-h)\bigg\}. \tag{22}$$

The terms of (22) that depend explicitly on the control u are obtained by applying the completion of square method as follows:

$$\bigg\{u(t)+\bar{R}^{-1}(t,\xi(t))\bigg[G_1(t,\xi(t))+G_2(t,\xi(t))x(t)$$
$$+G_3(t,\xi(t))x(t-h)+\int_t^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg]\bigg\}'$$
$$\bar{R}(t,\xi(t))\bigg\{u(t)+\bar{R}^{-1}(t,\xi(t))\bigg[G_1(t,\xi(t))+G_2(t,\xi(t))$$
$$\times x(t)+G_3(t,\xi(t))x(t-h)+\int_t^T G_4(t,\theta,\xi(t))$$
$$\times x(\theta-h)d\theta\bigg]\bigg\}-\bigg[G_1(t,\xi(t))+G_2(t,\xi(t))x(t)$$
$$+G_3(t,\xi(t))x(t-h)+\int_t^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg]'$$
$$\bar{R}^{-1}(t,\xi(t))\bigg[G_1(t,\xi(t))+G_2(t,\xi(t))x(t)+G_3(t,\xi(t))$$
$$\times x(t-h)+\int_t^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta\bigg],$$

Now, the cost functional $J_2(u(\cdot))$ can be written as:
$$J_2(u(\cdot))=v_2(T-h,x(T-h),\xi(T-h))+\mathbb{E}\bigg\{\int_{T-h}^T\bigg[u(t)$$
$$+\bar{R}^{-1}(t,\xi(t))\Big(G_1(t,\xi(t))+G_2(t,\xi(t))x(t)$$
$$+G_3(t,\xi(t))x(t-h)+\int_t^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta\Big)\bigg]'$$
$$\times\bar{R}(t,\xi(t))\bigg[u(t)+\bar{R}^{-1}(t,\xi(t))\Big(G_1(t,\xi(t))$$
$$+G_2(t,\xi(t))x(t)+G_3(t,\xi(t))x(t-h)+\int_{T-h}^T G_4(t,\theta,\xi(t))$$
$$\times x(\theta-h)d\theta\Big)\bigg]dt\bigg|\mathscr{F}(T-h)\bigg\}. \tag{23}$$

Note that $v_1(T-h,x(T-h),\xi(T-h))=v_2(T-h,x(T-h),\xi(T-h))$. From (23) and (21) it follows that for any $u(\cdot)\in\mathscr{A}$ we have:
$$J(u(\cdot))=v_1(0,x(0),i)+\mathbb{E}\bigg\{\int_0^{T-h}\bigg[u(t)+\underline{R}^{-1}(t,\xi(t)))$$

$$\times \left(\bar{G}_1(t,\xi(t)) + \bar{G}_2(t,\xi(t))x(t) + \bar{G}_3(t,\xi(t))x(t-h) \right.$$
$$\left. + \int_t^{t+h} \bar{G}_4(t,\theta,\xi(t))x(\theta-h)d\theta \right)' \underline{R}(t,\xi(t)) \left[u(t) \right.$$
$$+ \underline{R}^{-1}(t,\xi(t)) \left(\bar{G}_1(t,\xi(t)) + \bar{G}_2(t,\xi(t))x(t) \right.$$
$$+ \bar{G}_3(t,\xi(t))x(t-h) + \int_t^{t+h} \bar{G}_4(t,\theta,\xi(t))$$
$$\left. \left. \times x(\theta-h)d\theta \right) \right] dt \bigg| \xi(0) = i \right\} + \mathbb{E} \left\{ \mathbb{E} \left\{ \int_{T-h}^T \left[u(t) \right. \right. \right.$$
$$+ \bar{R}^{-1}(t,\xi(t)) \left(G_1(t,\xi(t)) + G_2(t,\xi(t)) \right.$$
$$\times x(t) + G_3(t,\xi(t))x(t-h) + \int_t^T G_4(t,\theta,\xi(t))$$
$$\left. \left. \times x(\theta-h)d\theta \right) \right]' \bar{R}(t,\xi(t)) \left[u(t) + \bar{R}^{-1}(t,\xi(t)) \right.$$
$$\left(G_1(t,\xi(t)) + G_2(t,\xi(t))x(t) + G_3(t,\xi(t))x(t-h) \right.$$
$$\left. \left. \left. + \int_{T-h}^T G_4(t,\theta,\xi(t))x(\theta-h)d\theta \right) \right] dt \bigg| \mathscr{F}(T-h) \right\} \right\}$$
$$\geq v_1(0,x(0),i).$$

This lower bound is achieved if and only if $u(t) = u^*(t)$ for *a.e.* $t \in [0,T]$ *a.s.*. $\qquad\square$

It is evident from the explicit closed-form of the optimal control law u^* that is in affine feedback form with respect to the system state, delayed system state, and a certain integral of past system state values. The coefficients in this control law are modulated by the Markov chain.

III. CONCLUSIONS

We have considered the optimal regulator problem for a linear stochastic system driven by additive and multiplicative noise that has a delay in its state, and the coefficients of which can randomly change regime based on a continuous-time Markov chain. The cost functional is a general version of the quadratic-linear criterion with Markovian switching coefficients. The solution is obtained in an explicit closed-form as an affine feedback on the system state, its delayed value, and an integral of its past values. Some interesting further problems to be considered are: the derivation of sufficient conditions under which the introduced system of coupled Riccati and partial differential equations are solvable (which, due to the coupling of equations, is a challenging problem even for systems without delay); the consideration of multiple-delays (as a generalisation to [2]); the inclusion of input-delays; and the applications to optimal investment problems (for which, see, e.g., [3], [8], [23], [9]).

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