Nonlinear Adaptive Differentiator Estimator For Ideal Signals

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Abstract— In this article, new nonlinear differentiator estimators for noise-free signals, with gains adaptively adjusted, are proposed. The algorithms allow finite time convergence of the differentiator estimation, and circumvent the unknown bounds of the second derivative of the original signal. Simulation results are led using different signal shapes to validate the efficiency of the proposed techniques, and to compare their performances to existing estimators depicted from literature.

I. INTRODUCTION

The problem of differentiator for signal measurements has a great impact on the implementation of most control techniques, and is still attracting attention from the research community. A variety of techniques have been proposed targeting exact and approximate derivative estimation error convergence, from ultimately bounded to asymptotic and finite time estimation [1], [2], [3].

In particular, the problem of differentiating a signal with bounded second derivative has been considered [1], [3]. An assumption dealing with the second derivative of the measured signal is the existence its upper bound (with known finite value). The uniform robust exact differentiator (URED) discussed in [2] is mainly based on such a condition. In [1], the authors have developed a finite time convergence exact differentiator (FTCED) to determine the first-order derivative of a given signal in case of unknown upper-bounded second-order derivative with a simple tuning process. The recommended differentiator with the employment of correction functions guarantees that the errors between the first-order derivative and its estimation converge to an arbitrarily small bounded region of the origin in finite time.

In this paper, we propose nonlinear estimator approaches, with adaptation gains, able to tend to the actual derivative of the noise-free signals with finite time convergence (FTC), under the more realistic condition of unknown upper-bound limit of the Lipschitz constant of the derivative. A basic nonlinear estimator with adaptation terms (NLEAT) and an augmented nonlinear estimator with adaptation and switching terms (NLEAST) will be discussed.

The paper is structured as follows. Section II introduces the considered problem of noise-free signal differentiation. Sections III and IV deal with the proposed NLEAT and NLEAST differentiators, respectively. We demonstrate the FTC of each estimator. We discuss the properties and requirements of the proposed design. In addition, the conditions required to select the parameters of the proposed estimator schemes, based on the stability and the convergence

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properties, are debrifed. Section V evaluate the performance of the new algorithms through simulations. Comparatives results are developed using two other methods depicted from literature. Finally, Section VI concludes the paper.

II. PROBLEM STATEMENT

Consider the problem of computing the derivative of a signal (function) $f : \mathbb{R}^+ \to \mathbb{R}$, based on a noise-free measurement and under the assumption of uniform bound L for the second derivative $|\ddot{f}(t)|$. f is differentiable, $\dot{f}$ is Lipschitz continuous on $\mathbb{R}^+$, and the second derivative, $\ddot{f}$, exists almost everywhere due to Rademacher's Theorem (this theorem says that a Lipschitz function on $\mathbb{R}$ is differentiable almost everywhere) [4], [5].

Given a measurable noise-free signal $f(t)$, estimate $\dot{f}(t)$ with $|\ddot{f}(t)| \leq L$; for unknown L. Set $\zeta_0 = f(t)$ and $\zeta_1 = \dot{f}(t)$. Then,

$$\dot{\zeta}_0 = \zeta_1 \tag{1a}$$
$$\dot{\zeta}_1 = \ddot{f} \tag{1b}$$

III. NLEAT DIFFERENTIATOR

To estimate the states ζ_0 and ζ_1, consider z_0 and z_1, and define the estimate errors $\sigma_0 = \zeta_0 - z_0$ and $\sigma_1 = \zeta_1 - z_1$, respectively. We propose the following estimator dynamics

$$\dot{z}_0 = k_0 \cdot \sigma_0 + \kappa(t) \cdot \mathsf{sgn}\sigma_0 + z_1 \tag{2a}$$
$$\dot{z}_1 = k_1 \cdot \sigma_0 + \beta\kappa(t) \cdot \mathsf{sgn}\sigma_0 \tag{2b}$$

where $k_0 > \beta$, k_1 are positive constant scalars, and $\kappa(t)$ is positive and monotonically non-decreasing gain given by

$$\dot{\kappa}(t) = \alpha \cdot |\sigma_0|, \quad \kappa(0) = \kappa_0 \tag{3}$$

with $\alpha > 0$ and $\kappa_0 \geq 0$.

From (1) and (2), we obtain the estimate error dynamics as

$$\begin{aligned}\dot{\sigma}_0 &= \dot{\zeta}_0 - \dot{z}_0 \\ &= -k_0\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + \sigma_1\end{aligned} \tag{4}$$

and

$$\begin{aligned}\dot{\sigma}_1 &= \dot{\zeta}_1 - \dot{z}_1 \\ &= \ddot{f} - k_1\sigma_0 - \beta\kappa(t)\mathsf{sgn}\sigma_0\end{aligned} \tag{5}$$

We define a new error state variable $s_1 = \sigma_1 - \beta\sigma_0$, to substitute σ_1. We obtain the error state dynamics, in σ_0 and s_1, as

$$\dot{\sigma}_0 = -(k_0 - \beta)\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + s_1 \tag{6}$$

and

$$\dot{s}_1 = \ddot{f} - (k_1 + \beta^2 - \beta k_0)\sigma_0 - \beta s_1 \tag{7}$$

Lemma 1: Consider the non-decreasing gain $\kappa(t)$ defined by (3). Assume $\sigma_0 \neq 0$. Given the positive Lipschitz constant L (s.t. $|\ddot{f}(t)| \leq L$), for any scalar $\gamma > 0$, there always exist a finite time $t^* \geq 0$ and $\lambda > 0$, s.t., for $t \geq t^*$

$$\kappa(t) \geq \lambda + \frac{L}{\gamma} \geq \lambda + \frac{|\ddot{f}|}{\gamma} \qquad (8)$$

where t^* represents the minimum compensating time, that is, when the adaptive gain $\kappa(t)$ compensates the upper-limit of the second derivative of the signal $f(t)$.

Proof: The proof of this statement is discussed through different scenarios.

Scenario No.1: Assume $|\sigma_0| \geq \bar{\sigma}_0 > 0$ for all $t \geq 0$, but not necessarily converging, then $\kappa(t) \geq \kappa_0 \geq 0$ is monotonically increasing. Thus, for any scalar $\gamma > 0$, there exist $t^* \geq 0$ and $\lambda > 0$, s.t., for $t \geq t^*$, (8) holds [6].

Scenario No.2: Assume the limit of $|\sigma_0|$, as $t \to \infty$, exists

$$\lim_{t \to \infty} |\sigma_0| = \bar{\sigma}_0 > 0 \qquad (9)$$

This limit $\bar{\sigma}_0$ could be either a finite positive value or the infinity. Then, there are positive scalars t_i and μ so that $|\sigma_0| > \mu > 0$ for all $t \geq t_i$. That being said,

$$\int_{t_i}^{\infty} |\sigma_0| d\tau = \lim_{t \to \infty} \int_{t_i}^{t} |\sigma_0| d\tau \geq \lim_{t \to \infty} (t - t_i)\mu = \infty \quad (10)$$

Considering (3), the positive scalar $\kappa(t) \geq \kappa_0$ is monotonically increasing, then for any scalar $\gamma > 0$, there exist $t^* \geq t_i$ and $\lambda > 0$, s.t., for $t \geq t^*$, (8) holds.

Scenario No.3: Assume $\forall t_i \geq 0$, $\exists \Delta t_i \neq 0$, $|\sigma_0| = 0$, s.t.,

$$0 \leq \int_{t_i}^{t_i + \Delta t_i} \alpha |\sigma_0| d\tau > 0 \qquad (11)$$

then,

$$\kappa(t) = \kappa(0) + \int_0^t \alpha |\sigma_0| d\tau \qquad (12)$$

is unbounded. In addition, from (6), for $t \in [t_i, t_i + \Delta t_i]$, $\dot{\sigma}_1 = \ddot{f} \neq 0$. That being said, σ_1 and s_1 vary. From (4) and (6), $\dot{\sigma}_0$ varies and makes $|\sigma_0| \neq 0$. Then, $\kappa(t)$ increases. This can repeatedly (continuously or intermittently) take place (within separate intervals of time) until $\kappa(t)$ reaches (or counterbalances) $|\ddot{f}|$ at $t \geq t^*$, for some $t^* > 0$. Thus, (8) holds.

Scenario No.4: Assume $|\sigma_0| \to 0$ as $t \to \infty$, or $\exists t_f \geq 0$, s.t., for any $\Delta t \neq 0$

$$0 \leq \int_{t_f}^{t_f + \Delta t} \alpha |\sigma_0| d\tau = 0 \qquad (13)$$

Then, $\forall t \geq t_f$, $|\sigma_0| = 0$. Note the latter is contradictory to Scenario 3. That being said, $\dot{\sigma}_0$ is bounded. From (6), s_1 and σ_1 are bounded. Two sub-cases can arise:

- If $\sigma_1 = 0$ and stays ultimately at zero, then the objective of the differentiator estimator algorithm is fulfilled (that is $z_1 \to \dot{f}(t)$ as $t \to \infty$).
- If $\sigma_1 \neq 0$, then from (4) and (6), $\dot{\sigma}_0$ varies and makes $|\sigma_0| \neq 0$, and $\kappa(t)$ is increasing. Thus, at $t \geq t^*$ for some $t^* > 0$, (8) holds.

In the following, we discuss the stability of the proposed estimator.

Proposition 1: Consider the state representation (1), of a noise-free measured signal $f(t)$, with its unknown derivative $\dot{f}(t)$ and bounded second-order derivative $|\ddot{f}(t)| \leq L$. The upper-bound limit L is unknown. Given positive scalars α, $k_0 > \beta$, k_1 and non-negative κ_0, the estimator dynamics (2), with the gain adaptation law (3), is converging to the origin in finite time.

Proof: Consider the Lyapunov-like function

$$V(\sigma_m) = \alpha_0 |\sigma_0| + \alpha_1 |s_1| \qquad (14)$$

along with the error dynamics (6) and (7), where $\sigma_m^T = [\sigma_0, s_1]$, α_0 and α_1 are positive scalars. We obtain

$$c_1 \|\sigma_m\| \leq V(\sigma_m) \leq c_2 \|\sigma_m\| \qquad (15)$$

with $c_1 = \min(\alpha_0, \alpha_1)$ and $c_2 = \max(\alpha_0, \alpha_1)$.

The global time derivative of $V(\sigma_m)$, for $\sigma_0 \neq 0$, satisfies

$$\begin{aligned}
\dot{V}(\sigma_m) &= \dot{\sigma}_0 \cdot \alpha_0 \mathsf{sgn}\sigma_0 + \alpha_1 \dot{s}_1 \cdot \mathsf{sgn}s_1 \\
&= \alpha_0 \Big(-(k_0 - \beta)\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + s_1 \Big)\mathsf{sgn}\sigma_0 + \\
&\quad \alpha_1 \Big(\ddot{f} - (k_1 + \beta^2 - \beta k_0)\sigma_0 - \beta s_1 \Big)\mathsf{sgn}s_1 \\
&\leq -\Big(\alpha_0(k_0 - \beta) - \alpha_1 |k_1 + \beta^2 - \beta k_0| \Big)|\sigma_0| \\
&\quad -(\alpha_1 \beta - \alpha_0)|s_1| - (\alpha_0 \kappa(t) - \alpha_1 |\ddot{f}|) \qquad (16)
\end{aligned}$$

From Lemma 1, we consider $\kappa(t) \geq \lambda + \dfrac{L}{\gamma}$, for $t \geq t^*$, and positive scalars $\gamma = \dfrac{\alpha_0}{\alpha_1}$ and λ. Then, given positive scalars α, β, k_0 and k_1, $\dot{V} < 0$, for $t \geq t^*$, if

$$\alpha_0(k_0 - \beta) \geq \alpha_1 |k_1 + \beta^2 - \beta k_0| \qquad (17)$$

and

$$\alpha_1 \beta \geq \alpha_0 \qquad (18)$$

In addition, if there exist positive scalars β_0 and β_1, s.t.,

$$\alpha_0(k_0 - \beta) - \alpha_1 |k_1 + \beta^2 - \beta k_0| \geq \beta_0 \qquad (19)$$

and

$$\alpha_1 \beta - \alpha_0 \geq \beta_1 \qquad (20)$$

then, using Lemma 1, for $t \geq t^*$,

$$\begin{aligned}
\dot{V}(t) &\leq -\beta_0 |\sigma_0| - \beta_1 |s_1| - \lambda \\
&\leq -c_3 \|\sigma_m\| - \lambda \qquad (21)
\end{aligned}$$

with $c_3 = \min(\beta_0, \beta_1)$.

Now, from (8) (refer to Lemma 1), (15) and (21), we obtain

$$\dot{V}(t) \leq -\frac{c_3}{c_2}V(t) - \lambda \qquad (22)$$

By using the comparison Lemma (Lemma 3.4 in [7]), we obtain $\forall t \geq t^*$

$$0 \leq V(t) \leq \left(V^* + \lambda \frac{c_2}{c_3}\right)e^{-\frac{c_3}{c_2}(t - t^*)} - \lambda \frac{c_2}{c_3} \qquad (23)$$

with $V^* = V(t^*)$. Thus, V is exponentially decreasing and reaches zero in finite time t_f that can be estimated as

$$t_f \leq \frac{c_2}{c_3} \ln\left(\frac{c_3 V^*}{c_2 \lambda} + 1\right) + t^* \tag{24}$$

From $V = \alpha_0|\sigma_0| + \alpha_1|s_1|$, we conclude that the states σ_0, s_1, and then, $\sigma_1 = s_1 + \beta\sigma_0$ are exponentially decreasing and reach zero in finite time.

Remarks 1: Let $\gamma = \frac{\alpha_0}{\alpha_1}$, $\delta = \frac{\beta_0}{\alpha_1}$ and $\nu = \frac{\beta_1}{\alpha_1}$ be positive scalars. From (19) and by choosing $k_0 > \beta$, we obtain

$$\delta + \left|k_1 - \beta(k_0 - \beta)\right| \leq \gamma(k_0 - \beta) \tag{25}$$

And, from (20), we obtain

$$\beta \geq \gamma + \nu \tag{26}$$

That is,

$$0 < (\beta - \gamma)(k_0 - \beta) + \delta \leq k_1 \leq (\beta + \gamma)(k_0 - \beta) - \delta \tag{27}$$

From (25), we notice that $\gamma(k_0 - \beta) \geq \delta > 0$. In conclusion, we have

$$0 < \frac{\delta}{k_0 - \beta} \leq \gamma \leq \beta - \nu \tag{28a}$$

$$(\beta - \gamma)(k_0 - \beta) + \delta \leq k_1 \leq (\beta + \gamma)(k_0 - \beta) - \delta \tag{28b}$$

The parameter selection of the proposed algorithm can be proceeded, from (28), as follows:

1) Set k_0 and β, *s.t.*, $k_0 > \beta > 0$.
2) Select γ, δ, ν, *s.t.*, $\frac{\delta}{k_0 - \beta} \leq \gamma \leq \beta - \nu$.
3) Set $k_1 \in \left[(\beta - \gamma)(k_0 - \beta) + \delta, (\beta + \gamma)(k_0 - \beta) - \delta\right]$.
4) Set $\alpha > 0$ and $\kappa_0 \geq 0$. α affects the rate of increase of the adaptive gain κ over time, and κ_0 is the initial and permanent component of $\kappa(t)$.

The Lyapunov parameters α_0, α_1, β_0 and β_1 can be obtained from $\gamma = \frac{\alpha_0}{\alpha_1}$, $\delta = \frac{\beta_0}{\alpha_1}$ and $\nu = \frac{\beta_1}{\alpha_1}$.

Remarks 2: As far as the non-decreasing gain $\kappa(t)$ remains below the compensating limit discussed in (8) (refer to Lemma 1), we have

$$\dot{V}(t) \leq -\beta_0|\sigma_0| - \beta_1|s_1| + \lambda^+$$
$$\leq -c_3\|\sigma_m\| + \lambda^+ \tag{29}$$

for λ^+ bounded positive scalar. Then, $\dot{V}(t) < 0$ for $\|\sigma_m\| > \frac{\lambda^+}{c_3}$. In this case, the estimate error dynamics is stable and ultimately uniformly bounded when (8) vanishes.

IV. NLEAST DIFFERENTIATOR

In this section, we modify the NLEAT algorithm by adding a switching term aimed at increasing the robustness and efficiency of the estimation process. Consider the estimator states z_0 and z_1, and define the estimate errors $\sigma_0 = \zeta_0 - z_0$

and $\sigma_1 = \zeta_1 - z_1$. We propose the following estimator dynamics

$$\dot{z}_0 = k_0 \cdot \sigma_0 + \kappa(t) \cdot \mathsf{sgn}\sigma_0 + z_1 \tag{30a}$$
$$\dot{z}_1 = k_1 \cdot \sigma_0 + (\beta\kappa(t) + k_2) \cdot \mathsf{sgn}\sigma_0 \tag{30b}$$

where $k_0 > \beta$, k_1 are positive constant scalars, k_2 non-negative, and $\kappa(t)$ is adaptively varying as (3), with $\alpha > 0$ and $\kappa_0 \geq 0$.

The dynamics (30) can be rewritten as

$$\dot{z}_0 = k_0 \cdot \sigma_0 + \hat{\kappa}(t) \cdot \mathsf{sgn}\sigma_0 + z_1 \tag{31a}$$
$$\dot{z}_1 = k_1 \cdot \sigma_0 + \breve{\kappa}(t) \cdot \mathsf{sgn}\sigma_0 \tag{31b}$$

where $\hat{\kappa}(t)$ is given by (3), with $\hat{\kappa}(0) = k_0$, and

$$\dot{\breve{\kappa}}(t) = \beta\alpha \cdot |\sigma_0| \tag{32}$$

with $\breve{\kappa}(0) = \beta\kappa_0 + k_2$.

From (1) and (30), we obtain the estimate error dynamics

$$\dot{\sigma}_0 = \dot{\zeta}_0 - \dot{z}_0$$
$$= -k_0\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + \sigma_1 \tag{33}$$

and

$$\dot{\sigma}_1 = \dot{\zeta}_1 - \dot{z}_1$$
$$= \ddot{f} - k_1\sigma_0 - (\beta\kappa(t) + k_2)\mathsf{sgn}\sigma_0 \tag{34}$$

By defining an alternative error state variable $s_1 = \sigma_1 - \beta\sigma_0$, to substitute σ_1, we obtain the error state dynamics, in σ_0 and s_1, as

$$\dot{\sigma}_0 = -(k_0 - \beta)\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + s_1 \tag{35}$$

and

$$\dot{s}_1 = \dot{\sigma}_1 - \beta\dot{\sigma}_0$$
$$= \ddot{f} - (k_1 + \beta^2 - \beta k_0)\sigma_0 - k_2\mathsf{sgn}\sigma_0 - \beta s_1 \tag{36}$$

In the following, we discuss the stability of the modified estimator.

Proposition 2: Consider the state representation (1) of measurable signal $f(t)$, with its unknown derivative $\dot{f}(t)$ and bounded second-order derivative, $|\ddot{f}(t)| \leq L$. The upper-bound limit L is unknown. Given positive scalars α, $k_0 > \beta > \gamma$, δ, ν, and non-negative scalars k_2 and κ_0, the estimator dynamics (30), with the gain adaptation law (3), is converging in finite time.

Proof: Consider the Lyapunov candidate function

$$V(\sigma_m) = \alpha_0|\sigma_0| + \alpha_1|s_1| \tag{37}$$

along with the error dynamics (6) and (7), where $\sigma_m^T = [\sigma_0, s_1]$, α_0 and α_1 are positive scalars. We obtain (15), with $c_1 = \min(\alpha_0, \alpha_1)$ and $c_2 = \max(\alpha_0, \alpha_1)$. The global

time derivative of $V(\sigma_m)$, for $\sigma_0 \neq 0$, satisfies

$$\dot{V}(\sigma_m) = \dot{\sigma}_0 \cdot \alpha_0 \mathsf{sgn}\sigma_0 + \alpha_1 \dot{s}_1 \cdot \mathsf{sgn}s_1$$
$$= \alpha_0 \Big(-(k_0 - \beta)\sigma_0 - \kappa(t)\mathsf{sgn}\sigma_0 + s_1 \Big)\mathsf{sgn}\sigma_0 +$$
$$\alpha_1 \Big(\ddot{f} - (k_1 + \beta^2 - \beta k_0)\sigma_0 - k_2\mathsf{sgn}\sigma_0 - \beta s_1 \Big)$$
$$\mathsf{sgn}s_1$$
$$\leq -\Big(\alpha_0(k_0 - \beta) - \alpha_1|k_1 + \beta^2 - \beta k_0| \Big)|\sigma_0|$$
$$- (\alpha_1\beta - \alpha_0)|s_1|$$
$$- \big(\alpha_0\kappa(t) - \alpha_1|\ddot{f}| - \alpha_1 k_2 \big) \qquad (38)$$

with $|\ddot{f}| \leq L$. From (3), the positive scalar $\kappa(t) \geq \kappa_0$ is monotonically increasing as long as $\sigma_0 \neq 0$. Using the principle of Lemma 1, for some positive scalars α_0, α_1 and non-negative scalar k_2, there exist $t^* \geq 0$ and $\lambda > 0$, *s.t.*, for $t \geq t^*$

$$\kappa(t) \geq \frac{\alpha_1}{\alpha_0}(L + k_2) + \lambda \geq \frac{\alpha_1}{\alpha_0}(|\ddot{f}| + k_2) + \lambda \qquad (39)$$

where t^* represents the minimum compensating time, that is, the time instant when the adaptive gain $\kappa(t^*)$ compensates for the amount $\frac{\alpha_1}{\alpha_0}(|\ddot{f}| + k_2)$ (refer to the right side of the inequality (39)).

The remaining steps of the current proof are similar to those elaborated in the proof of Proposition 1.

V. SIMULATIONS

To verify the efficacy of the proposed differentiators (*i.e.*, NLEAT and NLEAST), three different differentiable signals are used. The parameters of the proposed differentiator are chosen as $\alpha = 1.0$, $\kappa(0) = 0.0$, $k_0 = 15.0$, $k_1 = 10.0$, $k_2 = 20.0$ (in case of NLEAST algorithm), $\beta = 5.0$ and $\gamma = 4.5$. The obtained results are compared to those obtained with the FTCED and URED algorithms discussed in [1] and [2], respectively.

The dynamics of the URED are

$$\dot{z}_0 = k_0 \cdot \phi_0(\sigma_0) + z_1 \qquad (40a)$$
$$\dot{z}_1 = k_1 \cdot \phi_1(\sigma_0) \qquad (40b)$$

where z_0 and z_1 are the estimates of $\zeta_0 = f(t)$ and $\zeta_1 = \dot{f}(t)$, respectively, $\sigma_0 = \zeta_0 - z_0$, $k_0 = k_1 = 5$, and the correction functions $\phi_0(\sigma_0)$ and $\phi_1(\sigma_0)$ given by [1], [2]

$$\phi_0(\sigma_0) = |\sigma_0|^{\frac{1}{2}}\mathsf{sgn}(\sigma_0) + |\sigma_0|^{\frac{1}{2}}\sigma_0 \qquad (41a)$$
$$\phi_1(\sigma_0) = \frac{1}{2}\mathsf{sgn}(\sigma_0) + 2\sigma_0 + \frac{3}{2}|\sigma_0|\sigma_0 \qquad (41b)$$

The FTCED dynamics are similar to the URED, defined by (40), where its correction functions, $\phi_0(\sigma_0)$ and $\phi_1(\sigma_0)$, are given by [1]

$$\phi_0(\sigma_0) = \mu_0|\sigma_0|^{\gamma}\mathsf{sgn}(\sigma_0) + \mu_1|\sigma_0|^{\delta}\mathsf{sgn}(\sigma_0) + \mu_2\sigma_0 \quad (42a)$$
$$\phi_1(\sigma_0) = \phi_0'(\sigma_0)\phi_0(\sigma_0) \qquad (42b)$$

where μ_0, μ_1, and μ_2 are positive adjustable constants, γ and δ are positive constants satisfying $0 < \gamma < 1$ and $1 < \delta < 2$. For these simulations, we select $\mu_0 = 0.05$, $\mu_1 = 0.01$, $\mu_2 =$

10, $\gamma = \dfrac{2}{3}$, $\delta = \dfrac{3}{2}$, $k_0 = 2$ and $k_1 = 1$. $\phi_0'(\sigma_0) = \dfrac{\partial\phi_0(\sigma_0)}{\partial\sigma_0}$ denotes the partial derivative of $\phi_0(\sigma_0)$ with respect to σ_0, determined by

$$\phi_0'(\sigma_0) = \mu_0\gamma|\sigma_0|^{\gamma-1} + \mu_1\delta|\sigma_0|^{\delta-1}\sigma_0 + \mu_2 \qquad (43)$$

The initial conditions of all tested estimators are chosen as $z_0(0) = 5$ and $z_1(0) = 10$.

The following differentiated functions are tested:

$$f_1(t) = 5.3\sin\Big(\frac{\pi}{5}t + \frac{\pi}{4}\Big) + 2.7\sin\frac{\pi}{3}t \qquad (44a)$$
$$f_2(t) = 4.0\frac{t + 1.0}{t + 2.0}\sin 2t \qquad (44b)$$
$$f_3(t) = \big(-0.01t^3 + 0.12t^2 - 0.18t - 0.20 \big) \times$$
$$\sin\Big(2t + \frac{\pi}{3}\Big) \qquad (44c)$$

Note that $f_3(t)$ is beyond the class of signals mainly considered here and in most contributions, with its unbounded upper-limit second-derivative values. All the numerical tests are using a sampling period of $1\,\mathrm{msec}$. The estimation performances and estimation errors of the the comparative differentiators applied for the generated signals are illustrated in Figures 1–9. Peak values of the estimate errors (in % of the maximum absolute value of the actual derivative) for each signal, depicted during the steady-state regime, are shown in Table I. The achieved estimates are chattering-free for all considered and compared differentiators. It can be noted that the estimated derivative signals generated by the proposed differentiators (*i.e.*, NLEAT and NLEAST) show better results compared to the URED and FTCED methods. Considering initial conditions of the estimators different from the actual signal derivatives, one can note that proposed techniques generate the differentiation estimates with reasonably low overshoots. The effectiveness of the proposed techniques are justified by the level of accuracy depicted for the different tested signals. Indeed, the estimate error levels achieved by the proposed algorithms (in particular, the NLEAST) are smaller than those obtained using the URED and FTCED techniques (refer to Table I).
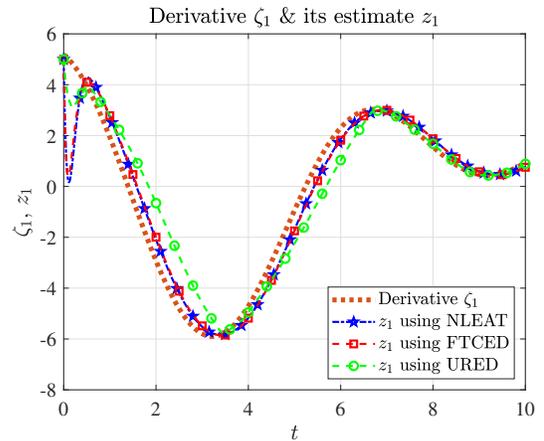

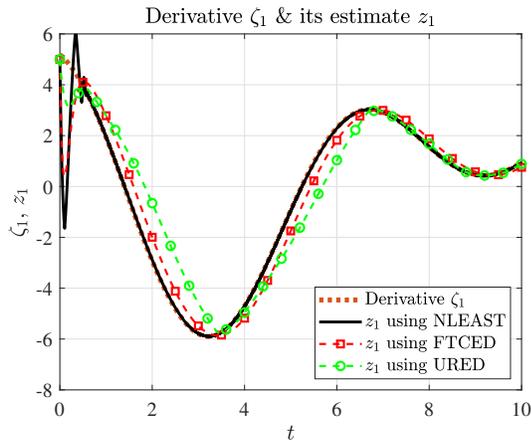Fig. 1. Derivative signal ζ_1 and its estimate z_1, for $f_1(t)$, using NLEAT, FTCED and URED techniques

Fig. 2. Derivative signal ζ_1 and its estimate z_1, for $f_1(t)$, using NLEAST, FTCED and URED techniques
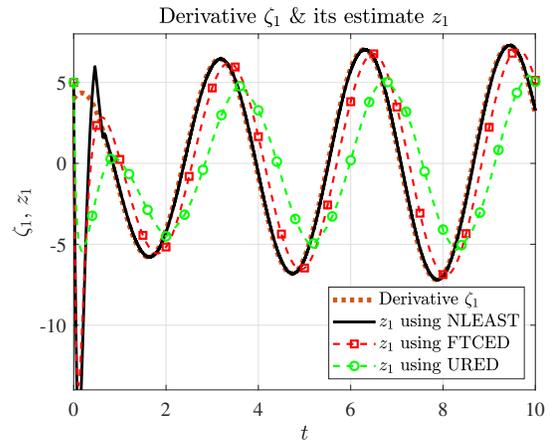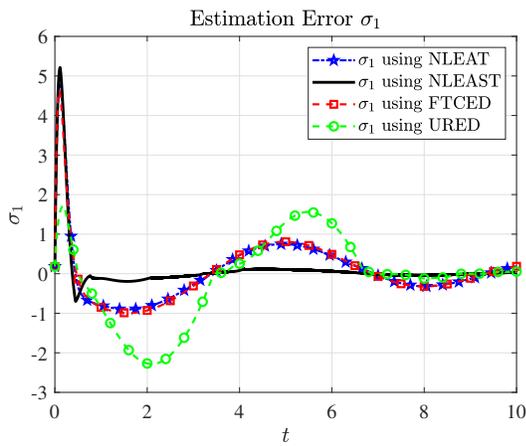

Fig. 3. Derivative estimate error σ_1, for $f_1(t)$, using NLEAT, NLEAST, FTCED and URED techniques
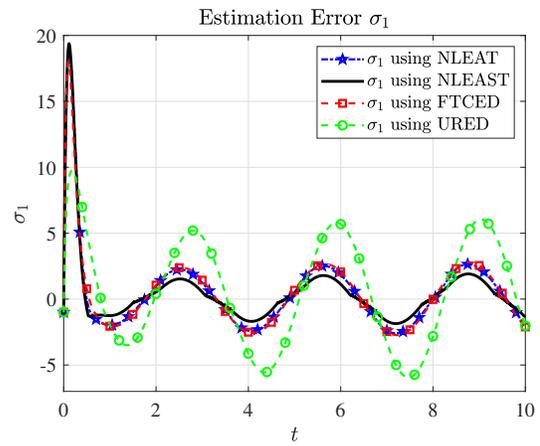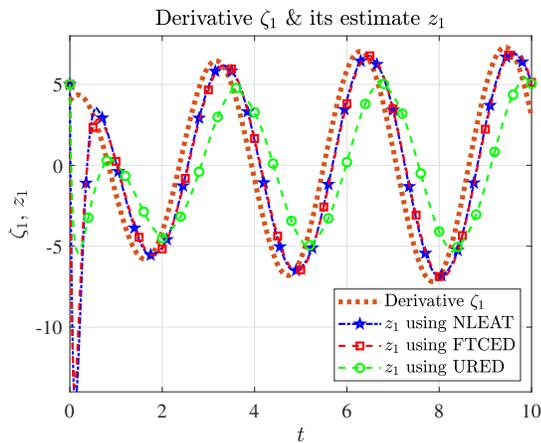

Fig. 4. Derivative signal ζ_1 and its estimate z_1, for $f_2(t)$, using NLEAT, FTCED and URED techniques

Fig. 5. Derivative signal ζ_1 and its estimate z_1, for $f_2(t)$, using NLEAST, FTCED and URED techniques

Fig. 6. Derivative estimate error σ_1, for $f_2(t)$, using NLEAT, NLEAST, FTCED and URED techniques
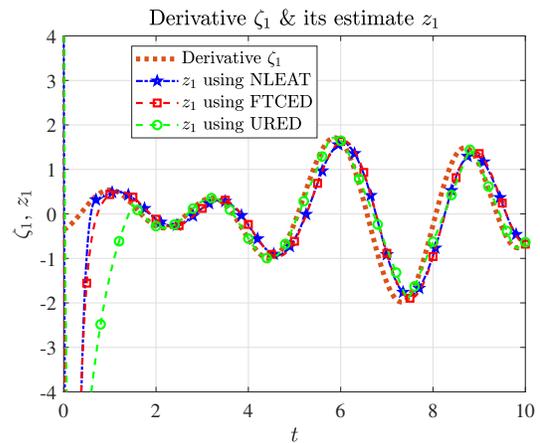

Fig. 7. Derivative signal ζ_1 and its estimate z_1, for $f_3(t)$, using NLEAT, FTCED and URED techniques

Note that the FTCED and URED guarantee that their estimate errors converge to an arbitrarily small bounded region of the origin in finite time, and asymptotically to

a compact set containing the origin, respectively [1], [2]. The FTCED estimates the first-order derivative of the given signal using a tuning process to bypass the knowledge of the upper-bounded second-order derivative. The parameters of the URED are selected depending on the upper bound of the second-order derivative which is assumed to be known. In practice, this upper bound value is unknown or rarely computed. The proposed methods are essentially based on adaptation gain laws to compensate for the uncertain terms of the error dynamics due to the unknown upper-bounded second-order derivative, and converge to the origin in finite time.
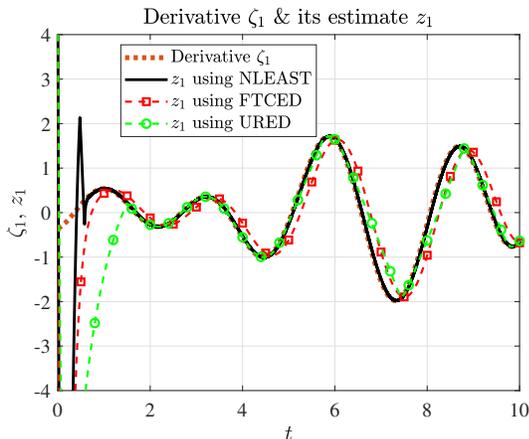

Fig. 8. Derivative signal ζ_1 and its estimate z_1, for $f_3(t)$, using NLEAST, FTCED and URED techniques

Remarks 3: Both NLEAT and NLEAST algorithms have been seen to lend themselves to parameter adaptation. By these methods, we intend to bridge the gap between theory and practice, potentially. In fact, the non-decreasing gains of the differentiator are constantly adjusting *w.r.t.* the characteristics of the signal to be differentiated independently of the unknown Lipschitz constant of the derivative. Moreover, these algorithms would bring more robustness *vis-à-vis* the noise levels in case of real signals. Despite the fact that, in this paper, the estimator targets the differentiation of pure (*chatter-free*) signals, the impact on the terms that could be affected by the level of noise (refer to switching components amplified by the adaptive gains) would be less significant within their dynamics playing a role of filters. These expectations would be addressed by assessing the estimator performance differentiating real measured signals.

VI. Conclusions

In this paper, we have proposed a new class of nonlinear signal differentiator design with adaptation laws taking into account the unknown upper-bound second-order derivative. These algorithms would be suitable for very demanding applications addressing the problem of unknown derivatives of the measured signals (often required for control purposes). To this end, detailed proofs are developed, and simulation results, including a comparative study to two existing methods, are carried out. A systematic and efficient parameter selection process to optimize the design performance can be followed appropriately. The simulations highlight the well-founded of the proposed differentiators. The validation of the proposed algorithms (in terms of both theoretic proof of stability and practical/numerical implementation), in the case of noisy measurements, is currently under investigation and relevant results will be revealed. Future works will be conducted on the discrete-time implementation of the differentiator, in particular in presence of noisy signals, and also on the extension of the proposed design to higher order differentiators.
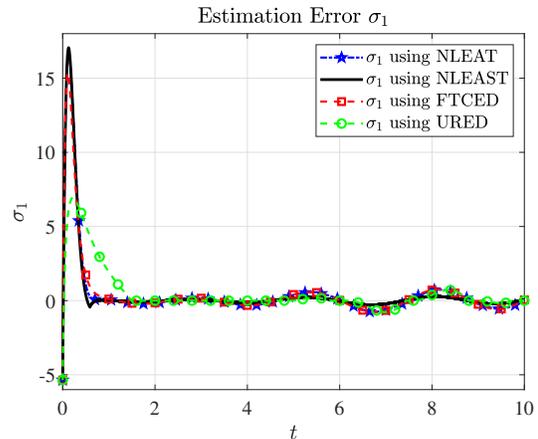

Fig. 9. Derivative estimate error σ_1, for $f_3(t)$, using NLEAT, NLEAST, FTCED and URED techniques

TABLE I

Maximum Estimate Errors (in % of Maximum Absolute Value of the Actual Derivative Signal) using the proposed LEAT, NLEAST and comparative techniques for noise free signals

Function	NLEAT	NLEAST	FTCED	URED
$f_1(t)$	4.1	0.2	4.4	11.5
$f_2(t)$	18.3	2.0	19.2	41.8
$f_3(t)$	20.4	2.4	20.8	20.8

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