On Convergent Smooth Differentiator Estimator For Ideal Signals

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Abstract— The objective of this paper is the design of smooth differentiator estimators for noise-free signals. A new class of differentiator estimators, with gains adaptively adjusted, is proposed. The algorithms allow ultimately uniformly bounded convergence of the estimators, and circumvent the unknown bounds of the second derivative of the original signal. Simulation results are led using different signal shapes to validate the efficiency of the proposed techniques, and to compare their performances to existing estimators depicted from literature.

I. INTRODUCTION

The problem of differentiator for signal measurements has a great impact on the implementation of most control techniques, and is still attracting attention from the research community. Recently, a variety of techniques have been proposed targeting exact and approximate derivative estimation error convergence, from ultimately bounded, asymptotic to finite-time estimation [1], [2], [3].

The problem of differentiating a signal with bounded second derivative has been considered [1], [3]. An assumption dealing with the second derivative of the measured signal is that its upper bound is finite and known constant. The uniform robust exact differentiator (URED) discussed in [2] is mainly based on such a condition. In [1], the authors have developed a finite-time continuous exact differentiator (FTCED) to determine the first-order derivative of a given signal, in case of unknown upper-bounded second-order derivative, supplemented by a tuning process. The recommended differentiator with the employment of correction functions guarantees that the errors between the first-order derivative and its estimation converge to an arbitrarily small bounded region of the origin in finite time. In [3], the authors have proposed some tuning paradigm by combining the Lipschitz constant bounds with a homogeneity-like scaling property of the global uniform convergence time. The differentiator's convergence time has been studied by extending the convergence time computation and the corresponding asymptotically exact bounds from [4].

In this paper, we propose simple, smooth and almost linear estimator approaches, with adaptation gains, able to converge to the actual derivative of the noise-free signals with an ultimately uniformly bounded (UUB) error, under the more realistic condition of unknown upper-bound limit of the Lipschitz constant of the derivative. A basic linear estimator with adaptation terms (LEAT) and an augmented linear estimator with adaptation and switching terms (LEAST) will be discussed.

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The paper is structured as follows. Section II introduces the considered problem of signal differentiation. In Sections III and IV, we discuss the proposed LEAT and LEAST differentiators. We demonstrate the UUB error convergence of the estimators. Section V evaluates the performance of the proposed estimators through simulations. Comparative results are developed using two other methods depicted from literature. Finally, Section VI concludes the paper.

II. PROBLEM STATEMENT

Consider the problem of computing the derivative of a signal (function) $f : \mathbb{R}^+ \to \mathbb{R}$, based on a noise-free measurement and under the assumption of uniform bound L for the second derivative $|\ddot{f}(t)|$. f is differentiable, $\dot{f}$ is Lipschitz continuous on $\mathbb{R}^+$, and the second derivative, $\ddot{f}$, exists almost everywhere due to Rademacher's Theorem (this theorem says that a Lipschitz function on $\mathbb{R}$ is differentiable almost everywhere) [5], [6].

Given a measurable noise-free signal $f(t)$, estimate $\dot{f}(t)$ with $|\ddot{f}(t)| \leq L$; for unknown L. Set $\zeta_0 = f(t)$ and $\zeta_1 = \dot{f}(t)$. Then,

$$\dot{\zeta}_0 = \zeta_1 \tag{1a}$$
$$\dot{\zeta}_1 = \ddot{f} \tag{1b}$$

III. LEAT DIFFERENTIATOR

Consider the estimator states z_0 and z_1, and define the estimate errors $\sigma_0 = \zeta_0 - z_0$ and $\sigma_1 = \zeta_1 - z_1$. We propose the following estimator dynamics

$$\dot{z}_0 = \big(k_0 + \kappa(t)\big) \cdot \sigma_0 + z_1 \tag{2a}$$
$$\dot{z}_1 = \big(k_1 + \beta\kappa(t)\big) \cdot \sigma_0 \tag{2b}$$

where β, $k_0 > \beta$ and k_1 are positive constant scalars. k_0 and k_1 represent permanent gains, and $\kappa(t)$ is adaptively varying as

$$\dot{\kappa}(t) = \alpha \cdot |\sigma_0| \tag{3}$$

with $\alpha > 0$. Without loss of generality, set $\kappa(0) = \kappa_0 = 0$. Note that the time varying gain $\kappa(t)$ is positive and monotonically increasing as long as $\sigma_0 \neq 0$.

The dynamics (2) can be rewritten as

$$\dot{z}_0 = \hat{\kappa}(t) \cdot \sigma_0 + z_1 \tag{4a}$$
$$\dot{z}_1 = \check{\kappa}(t) \cdot \sigma_0 \tag{4b}$$

where $\hat{\kappa}(t)$ is given by (3) with $\hat{\kappa}(0) = k_0$, and

$$\dot{\check{\kappa}}(t) = \beta\alpha \cdot |\sigma_0| \tag{5}$$

with $\check{\kappa}(0) = k_1 + \beta k_0$.

From (1) and (2), we obtain the estimate error dynamics, as

$$\dot{\sigma}_0 = \dot{\zeta}_0 - \dot{z}_0$$
$$= -\big(k_0 + \kappa(t)\big)\sigma_0 + \sigma_1 \qquad (6)$$

and

$$\dot{\sigma}_1 = \dot{\zeta}_1 - \dot{z}_1$$
$$= \ddot{f} - \big(k_1 + \beta\kappa(t)\big)\sigma_0 \qquad (7)$$

We define a new error state variable $s_1 = \sigma_1 - \beta\sigma_0$ to substitute σ_1. We obtain the error state dynamics, in σ_0 and s_1, as

$$\dot{\sigma}_0 = -\big(k_0 - \beta + \kappa(t)\big)\sigma_0 + s_1 \qquad (8)$$

and

$$\dot{s}_1 = \ddot{f} - \big(k_1 + \beta^2 - \beta k_0\big)\sigma_0 - \beta s_1 \qquad (9)$$

In the following, we discuss the stability of the proposed estimator.

Proposition 1: Consider the state representation (1), of a measurable signal $f(t)$, with its unknown derivative $\dot{f}(t)$ and bounded second-order derivative $|\ddot{f}(t)| \leq L$. The upper-bound limit L is unknown. Given positive scalars α, β, k_0, and k_1, the estimator dynamics (2), with the gain adaptation law (3), is UUB.

Proof: Consider the Lyapunov-like function

$$V(\sigma_m) = \alpha_0|\sigma_0| + \alpha_1|s_1| \qquad (10)$$

along with the error dynamics (8) and (9), where $\sigma_m^T = [\sigma_0, s_1]$, α_0 and α_1 are positive scalars. We obtain

$$c_1\|\sigma_m\| \leq V(\sigma_m) \leq c_2\|\sigma_m\| \qquad (11)$$

with $c_1 = \min(\alpha_0, \alpha_1)$ and $c_2 = \max(\alpha_0, \alpha_1)$. Using $|\ddot{f}(t)| \leq L$, along with the error dynamics (8) and (9), the global time derivative of $V(\sigma_m)$ satisfies, for $\sigma_0 \neq 0$,

$$\dot{V}(\sigma_m) = \alpha_0\dot{\sigma}_0 \cdot \mathsf{sgn}\sigma_0 + \alpha_1\dot{s}_1 \cdot \mathsf{sgn}s_1$$
$$\leq -\Big(\alpha_0\big(k_0 - \beta\big) - \alpha_1\big|k_1 + \beta^2 - \beta k_0\big|\Big)|\sigma_0|$$
$$- \big(\alpha_1\beta - \alpha_0\big)|s_1| - \big(\alpha_0\kappa(t)|\sigma_0| - \alpha_1 L\big) \quad (12)$$

Given positive scalars α, β, k_0 and k_1, and considering the time varying positive gain $\kappa(t)$; which is monotonically increasing as long as $\sigma_0 \neq 0$; if

$$\alpha_0\big(k_0 - \beta\big) - \alpha_1\big|k_1 + \beta^2 - \beta k_0\big| > 0 \qquad (13)$$

and

$$\alpha_1\beta - \alpha_0 > 0 \qquad (14)$$

then $\dot{V} < 0$, for $|\sigma_0| \geq \dfrac{\alpha_1 L}{\alpha_0\kappa(t)}$. For $\sigma_0 \neq 0$, the lower limit term $\dfrac{\alpha_1 L}{\alpha_0\kappa(t)}$ is decreasing. That being said, the domain of stability of σ_0 and s_1 (and of $\sigma_1 = s_1 + \beta\sigma_0$, systematically) does not contain the origin, but its lower limit on σ_0-axis is approaching the origin. The error dynamics is UUB.

Remarks 1: Consider $\dfrac{\alpha_0}{\alpha_1} = \gamma$, for any $\gamma > 0$. Combining (13) and (14) with $k_0 > \beta$, we obtain

$$k_0 > \beta > \gamma \qquad (15)$$

and

$$\big(\beta - \gamma\big)\big(k_0 - \beta\big) \leq k_1 < \big(\beta + \gamma\big)\big(k_0 - \beta\big) \qquad (16)$$

From (15) and (16), the gain selection procedure can be set as follows:

1) Set k_0 and β, s.t., $k_0 > \beta > 0$.
2) Select γ, s.t., $0 < \gamma < \beta$. Note that α_0 and α_1 can be selected freely from $\gamma = \dfrac{\alpha_0}{\alpha_1}$.
3) Set $k_1 \in \Big[\big(\beta - \gamma\big)\big(k_0 - \beta\big), \big(\beta + \gamma\big)\big(k_0 - \beta\big)\Big]$.
4) Set $\alpha > 0$ and $\kappa_0 \geq 0$. α affects the rate of increase of the gain κ over time. Without loss of generality, set $\kappa(0) = \kappa_0 = 0$. Any nonzero value of the initial gain κ_0 contributes into the permanent constant gains k_0 and k_1 in (2).

It is clear, from the procedure shown above, that the estimator parameters; in particular α and β; can be freely selected. Then, k_0 and k_1 are obtained from the conditions (15) and (16).

IV. LEAST DIFFERENTIATOR

In this section, we modify the LEAT algorithm by adding a nonlinear switching term that tends to increase the robustness and the efficiency of the estimation process. We propose the following estimator dynamics

$$\dot{z}_0 = \big(k_0 + \kappa(t)\big) \cdot \sigma_0 + z_1 \qquad (17a)$$
$$\dot{z}_1 = \big(k_1 + \beta\kappa(t)\big) \cdot \sigma_0 + k_2 \cdot \mathsf{sgn}\sigma_0 \qquad (17b)$$

where $k_0 > \beta$, k_1 are positive constant scalars, and $\kappa(t)$ is given by (3) with $\kappa(0) = 0$. From (1) and (17), we obtain the estimate error dynamics as

$$\dot{\sigma}_0 = -\big(k_0 + \kappa(t)\big)\sigma_0 + \sigma_1 \qquad (18)$$

and

$$\dot{\sigma}_1 = \ddot{f} - \big(k_1 + \beta\kappa(t)\big)\sigma_0 - k_2 \cdot \mathsf{sgn}\sigma_0 \qquad (19)$$

Using the change of variable $s_1 = \sigma_1 - \beta\sigma_0$, we obtain the error state dynamics, in σ_0 and s_1, as

$$\dot{\sigma}_0 = -\big(k_0 - \beta + \kappa(t)\big)\sigma_0 + s_1 \qquad (20)$$

and

$$\dot{s}_1 = \ddot{f} - \big(k_1 + \beta^2 - \beta k_0\big)\sigma_0 - k_2\mathsf{sgn}\sigma_0 - \beta s_1 \qquad (21)$$

In the following, we discuss the stability of the modified estimator.

Proposition 2: Consider the state representation (1), of measurable signal $f(t)$, with its unknown derivative $\dot{f}(t)$ and bounded second-order derivative $|\ddot{f}(t)| \leq L$. The upper-bound limit L is unknown. Given positive scalars α, β, k_0, and k_1, the estimator dynamics (17), with the gain adaptation law (3), is UUB.

Proof: Consider the Lyapunov candidate function (10) along with the error dynamics (20) and (21), where $\sigma_m^T = [\sigma_0, s_1]$, α_0 and α_1 are positive scalars. We obtain (11). The global time derivative of $V(\sigma_m)$, for $\sigma_0 \neq 0$, satisfies

$$
\begin{aligned}
\dot{V}(\sigma_m) &= \alpha_0 \dot{\sigma}_0 \cdot \mathsf{sgn}\sigma_0 + \alpha_1 \dot{s}_1 \cdot \mathsf{sgn}s_1 \\
&\leq - \Big(\alpha_0\big(k_0 - \beta\big) - \alpha_1\big|k_1 + \beta^2 - \beta k_0\big|\Big)|\sigma_0| \\
&\quad - \big(\alpha_1\beta - \alpha_0\big)|s_1| - \big(\alpha_0\kappa(t)|\sigma_0| \\
&\quad - \alpha_1 L - k_2\big)
\end{aligned}
\tag{22}
$$

Given positive scalars α, β, k_0 and k_1, if

$$
\alpha_0\big(k_0 - \beta\big) - \alpha_1\big|k_1 + \beta^2 - \beta k_0\big| > 0 \tag{23}
$$

and

$$
\alpha_1\beta - \alpha_0 > 0 \tag{24}
$$

for positive scalars β_0 and β_1, then $\dot{V} < 0$, for $|\sigma_0| \geq \dfrac{\alpha_1 L + k_2}{\alpha_0\kappa(t)}$. That being said, the estimate error dynamics is UUB.

Remarks 2: The domain of stability of σ_0 and s_1 (and of $\sigma_1 = s_1 + \beta\sigma_0$, eventually) does not contain the origin, but its lower limit on σ_0-axis is approaching the origin, and is smaller than the one obtained with the LEAT differentiator. This would contribute in more robust convergence of the estimator dynamics.

Remarks 3: Note that, the FTCED and URED guarantee that their estimate errors converge to an arbitrarily small bounded region of the origin in finite time, and to a compact set containing the origin, respectively. The parameters of the URED are selected depending on the upper bound of the second-order derivative which is assumed to be known. In practice, this upper bound value is unknown or rarely computed. The FTCED estimated the first-order derivative of the given signal using a tuning process to bypass the knowledge of the upper-bounded second-order derivative. In this paper, the proposed methods are based on adaptation gain laws to compensate for the uncertain terms of the error dynamics due to the unknown upper-bounded second-order derivative.

V. Simulations

To verify the efficacy of the proposed differentiators (*i.e.*, LEAT and LEAST), three different differentiable signals are used. The parameters of the proposed differentiator are chosen as $\alpha = 1.0$, $\kappa(0) = 0.0$, $k_0 = 15.0$, $k_1 = 10.0$, $k_2 = 20.0$ (in case of LEAST algorithm), $\beta = 5.0$ and $\gamma = 4.5$.

The obtained results are compared to those obtained with the FTCED and URED algorithms discussed in [1] and [2], respectively.

The dynamics of the URED are

$$
\dot{z}_0 = k_0 \cdot \phi_0(\sigma_0) + z_1 \tag{25a}
$$
$$
\dot{z}_1 = k_1 \cdot \phi_1(\sigma_0) \tag{25b}
$$

where z_0 and z_1 are the estimates of $\zeta_0 = f(t)$ and $\zeta_1 = \dot{f}(t)$, respectively, $\sigma_0 = \zeta_0 - z_0$, $k_0 = k_1 = 5$, and the correction functions $\phi_0(\sigma_0)$ and $\phi_1(\sigma_0)$ given by [1], [2]

$$
\phi_0(\sigma_0) = |\sigma_0|^{\frac{1}{2}}\mathsf{sgn}(\sigma_0) + |\sigma_0|^{\frac{1}{2}}\sigma_0 \tag{26a}
$$
$$
\phi_1(\sigma_0) = \frac{1}{2}\mathsf{sgn}(\sigma_0) + 2\sigma_0 + \frac{3}{2}|\sigma_0|\sigma_0 \tag{26b}
$$

The FTCED dynamics are similar to the URED, defined by (25), where its correction functions, $\phi_0(\sigma_0)$ and $\phi_1(\sigma_0)$, are given by [1]

$$
\phi_0(\sigma_0) = \mu_0|\sigma_0|^\gamma\mathsf{sgn}(\sigma_0) + \mu_1|\sigma_0|^\delta\mathsf{sgn}(\sigma_0) + \mu_2\sigma_0 \tag{27a}
$$
$$
\phi_1(\sigma_0) = \phi_0'(\sigma_0)\phi_0(\sigma_0) \tag{27b}
$$

where μ_0, μ_1, and μ_2 are positive adjustable constants, γ and δ are positive constants satisfying $0 < \gamma < 1$ and $1 < \delta < 2$. For these simulations, we select $\mu_0 = 0.05$, $\mu_1 = 0.01$, $\mu_2 = 10$, $\gamma = \frac{2}{3}$, $\delta = \frac{3}{2}$, $k_0 = 2$ and $k_1 = 1$. $\phi_0'(\sigma_0) = \dfrac{\partial\phi_0(\sigma_0)}{\partial\sigma_0}$ denotes the partial derivative of $\phi_0(\sigma_0)$ with respect to σ_0, determined by

$$
\phi_0'(\sigma_0) = \mu_0\gamma|\sigma_0|^{\gamma-1} + \mu_1\delta|\sigma_0|^{\delta-1}\sigma_0 + \mu_2 \tag{28}
$$

The initial conditions of all tested estimators are chosen as $z_0(0) = 5$ and $z_1(0) = 10$.

The following differentiated functions are tested:

$$
f_1(t) = 5.3\sin\big(\frac{\pi}{5}t + \frac{\pi}{4}\big) + 2.7\sin\frac{\pi}{3}t + 2.5t + 3.5 \tag{29a}
$$
$$
\begin{aligned}
f_2(t) &= \int_0^t \Big(\int_0^\tau \int_0^\nu 21.3R_N(\eta)d\eta d\nu + 4.5\Big)d\tau, \\
&\quad \text{with } R_N \text{ a normally (Gaussian) distributed} \\
&\quad \text{random signal of 0.0 mean and 7.5 average}
\end{aligned}
\tag{29b}
$$
$$
\begin{aligned}
f_3(t) &= \big(-0.01t^3 + 0.12t^2 - 0.18t - 0.20\big)\times \\
&\quad \sin\big(2t + \frac{\pi}{3}\big)
\end{aligned}
\tag{29c}
$$

Note that $f_3(t)$ is beyond the class of signals mainly considered here and in most contributions, with its unbounded upper-limit second-derivative values. All the numerical tests are implemented using a sampling period of $1\,\text{msec}$. The estimation performances and estimation errors of the comparative differentiators applied for the generated signals are illustrated in Figures 1–15. Peak values of the derivative estimation errors (in % of the maximum absolute value of the actual derivative) for each signal, depicted during the steady-state regime, are shown in Table I.

The achieved estimates are chattering-free for all considered and compared differentiators. It can be noted that the estimated derivative generated by the proposed differentiators (*i.e.*, LEAT and LEAST) can reach the actual one in a shorter time compared to the URED and FTCED methods. The effectiveness of the proposed techniques are justified by the level of accuracy depicted for the different tested signals. Indeed, the estimate error levels achieved by the proposed algorithms are smaller than those obtained using the URED and FTCED techniques.
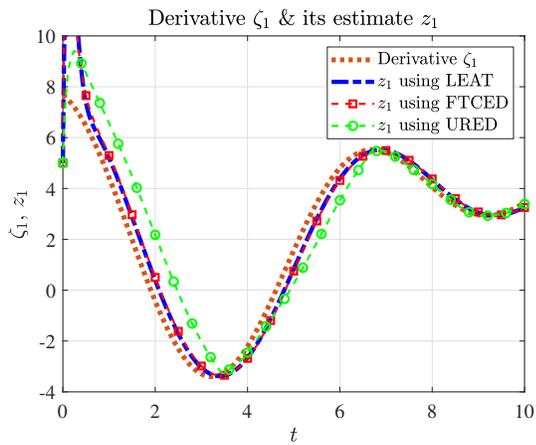
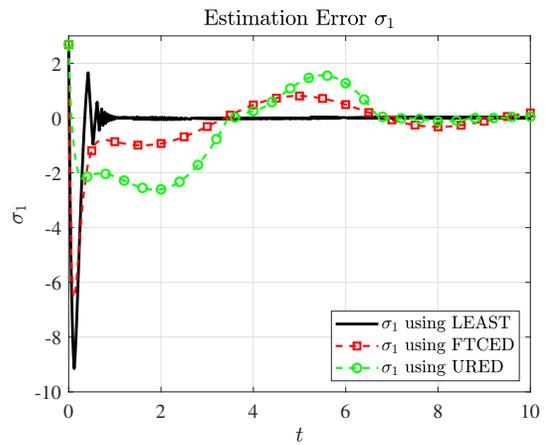Fig. 1. Derivative signal ζ_1 and its estimate z_1, for $f_1(t)$, using LEAT, FTCED and URED techniques

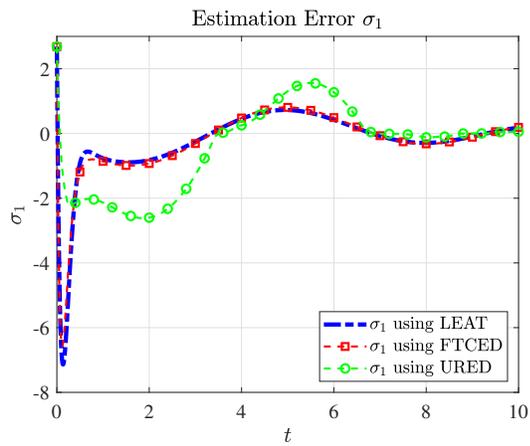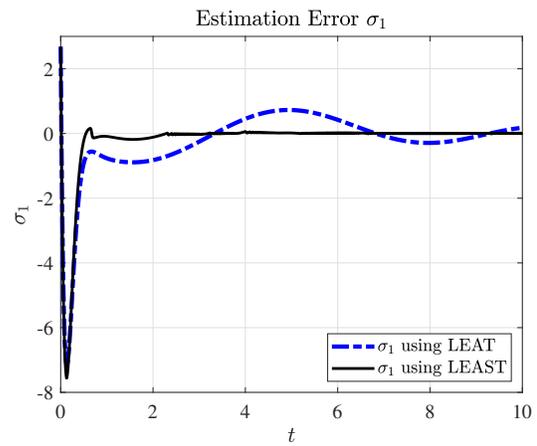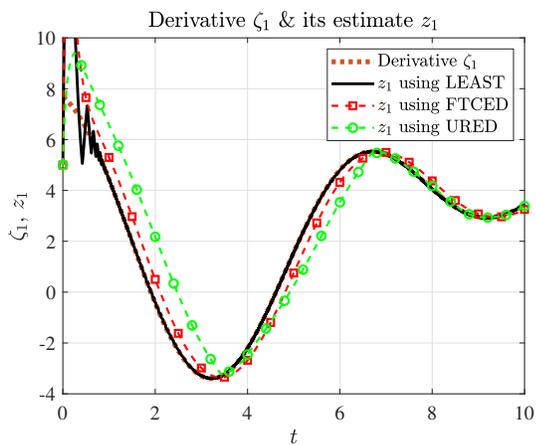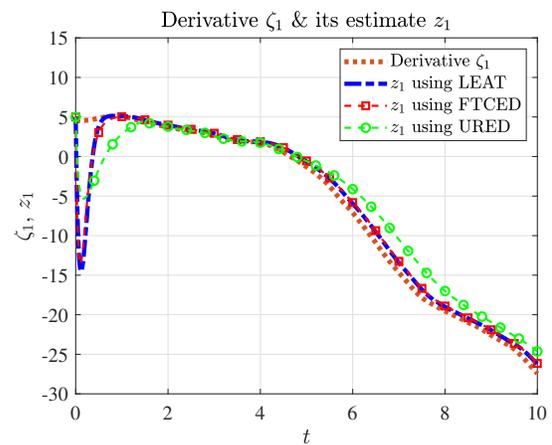Fig. 2. Derivative estimate error σ_1, for $f_1(t)$, using LEAT, FTCED and URED techniques

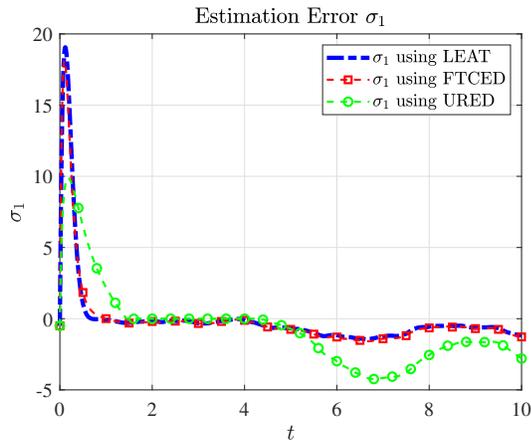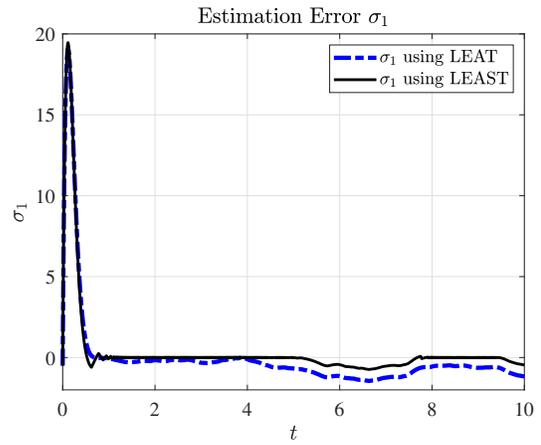Fig. 3. Derivative signal ζ_1 and its estimate z_1, for $f_1(t)$, using LEAST, FTCED and URED techniques

Fig. 4. Derivative estimate error σ_1, for $f_1(t)$, using LEAST, FTCED and URED techniques

Fig. 5. Derivative estimate error σ_1, for $f_1(t)$, using LEAT and LEAST techniques

Fig. 6. Derivative signal ζ_1 and its estimate z_1, for $f_2(t)$, using LEAT, FTCED and URED techniques

Fig. 7. Derivative estimate error σ_1, for $f_2(t)$, using LEAT, FTCED and URED techniques

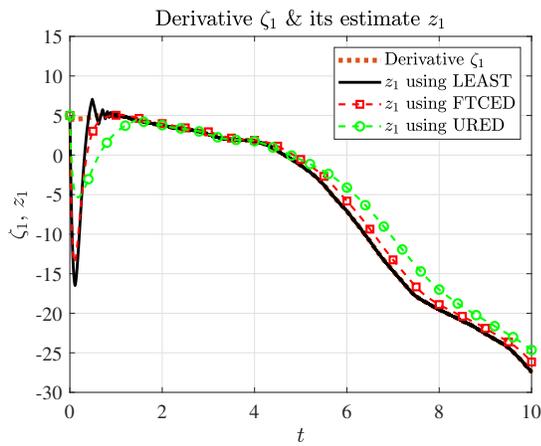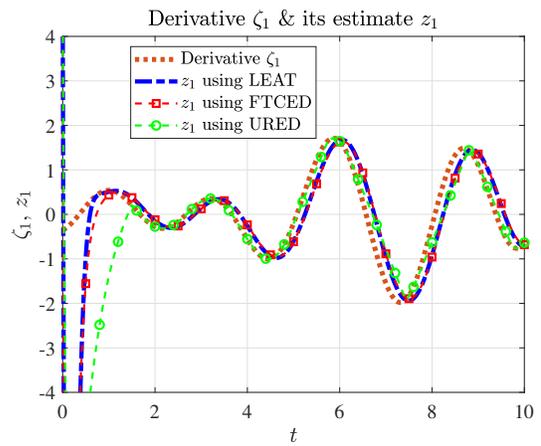Fig. 10. Derivative estimate error σ_1, for $f_2(t)$, using LEAT and LEAST techniques

Fig. 8. Derivative signal ζ_1 and its estimate z_1, for $f_2(t)$, using LEAST, FTCED and URED techniques

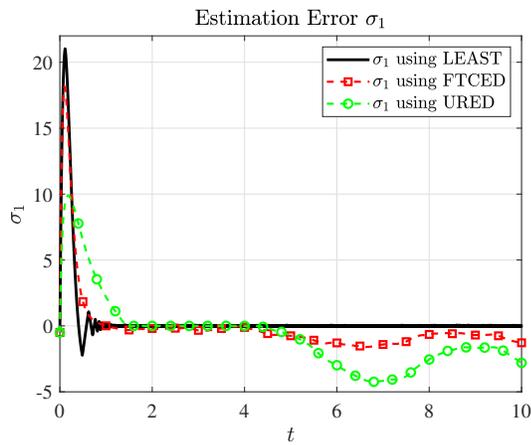Fig. 11. Derivative signal ζ_1 and its estimate z_1, for $f_3(t)$, using LEAT, FTCED and URED techniques

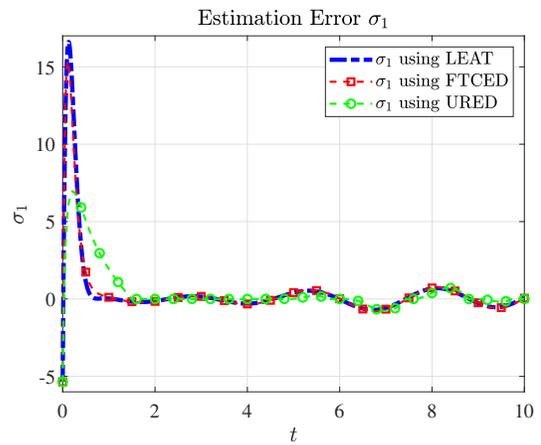Fig. 9. Derivative estimate error σ_1, for $f_2(t)$, using LEAST, FTCED and URED techniques

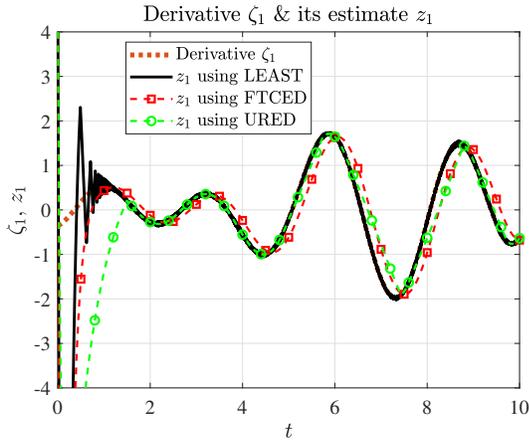Fig. 12. Derivative estimate error σ_1, for $f_3(t)$, using LEAT, FTCED and URED techniques

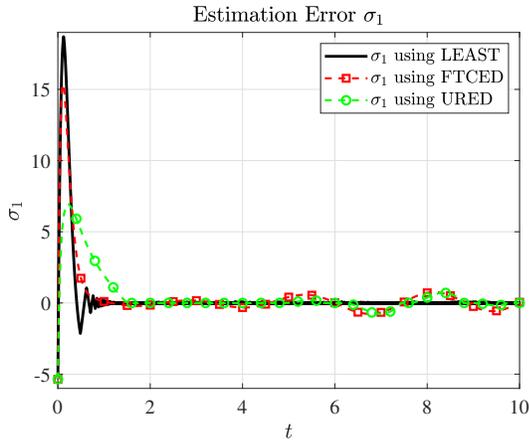Fig. 13. Derivative signal ζ_1 and its estimate z_1, for $f_3(t)$, using LEAST, FTCED and URED techniques

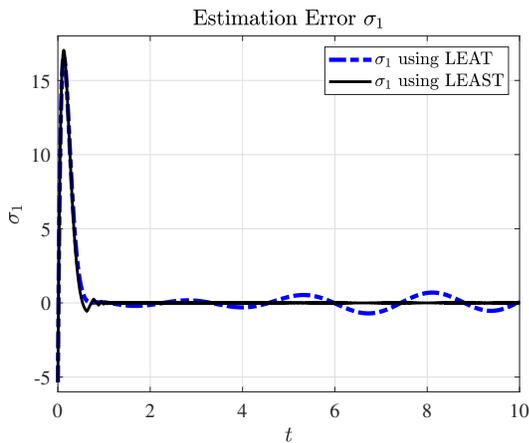Fig. 14. Derivative estimate error σ_1, for $f_3(t)$, using LEAST, FTCED and URED techniques

Fig. 15. Derivative estimate error σ_1, for $f_3(t)$, using LEAT and LEAST techniques

Note that the URED and FTCED algorithms, by using appropriate correction functions, guarantee that the estimation errors converge to an arbitrarily small bounded region

of the origin in finite time in [1], and in asymptotic order when the Lipschitz constant of its derivative is bounded by a known constant in [2]. The proposed estimators use simpler and quasi-linear descriptors to guarantee similar type of convergence under a relaxed condition of unknown upper-bound limit of the Lipschitz constant of the derivative. As shown in Figures (5), (10), (15) and Table I, the augmented switching term of the LEAST differentiator brings more robustness, in terms of accuracy, convergence rate and type (switching performance about the origin) compared to the LEAT one. The LEAST has demonstrated the best performance in case of unbounded Lipschitz condition of the derivative.

TABLE I
MAXIMUM ESTIMATE ERRORS (IN % OF MAXIMUM ABSOLUTE VALUE OF THE ACTUAL DERIVATIVE SIGNAL) USING THE PROPOSED LEAT, LEAST AND COMPARATIVE TECHNIQUES FOR NOISE FREE SIGNALS

Function	LEAT	LEAST	FTCED	URED
$f_1(t)$	3.8	0.2	4.4	11.5
$f_2(t)$	4.4	0.2	4.9	13.0
$f_3(t)$	19.1	1.7	20.8	20.8

VI. CONCLUSIONS

In this paper, we have proposed a new class of signal differentiator design with adaptation laws taking into account the unknown upper-bound second-order derivative. These estimators are suitable for the applications addressing the problem of unknown derivatives of the measured signals (*e.g.*, control problems) . To this end, detailed proofs are discussed and simulation results, including a comparative study with two existing methods, are performed. These simulations highlight the well-founded of the proposed differentiator. The class of differentiators considered in this paper features a simpler form and an easier design compared to the existing methods. It can be employed for numeric differentiation and real-time control systems with small noises. Future works will be conducted on the implementation of the differentiator in presence of noisy signals, and also the extension of the proposed design to higher order differentiators.

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