

Topological Roughness of the Phase Space of Dynamic Systems*

Abstract — The main provisions of the method of research the roughness of dynamic systems, based on the concept of coarseness according to Andronov-Pontryagin and called the "topological roughness method," are considered. Definitions of concepts of maximum roughness and minimum non-roughness of dynamic systems are given. The corresponding theorems are formulated on the necessary and sufficient conditions for the reachability of maximum roughness and minimum non-roughness, as well as the emergence of bifurcations of topological structures of the phase space of dynamic systems, which were proved in the author's works. The method allows you to control the roughness of control systems based on the theorem formulated using the Sylvester matrix equation. The method can be used for studies of roughness and bifurcations of dynamic systems, as well as synergetic systems and chaos of various physical nature. In the author's works, the method has been tested for studying the synergetic systems of Lorenz, Rössler, Belousov-Zhabotinsky, "predator-prey", Chua, Rikitake dynamo, Henon maps, Hopf bifurcations, and models of economic systems of the Schumpeter and Kaldor types.

I. Introduction

Problems of investigation of roughness of dynamic systems, evaluation of robustness and synthesis of roughness (robustness) control systems are paid great attention in modern theory of dynamic systems and theory of control [1-7].

In the theory of dynamic systems, there are two different approaches to the problem of roughness: 1) based on the concept of Peixoto roughness or otherwise "structural stability"; 2) on the basis of the concept of roughness according to Andronov - Pontryagin, when, unlike the previous one, it is required ε - the proximity of the original and perturbed homeomorphisms [1, 2, 8].

In the work [9] on the basis of the concept of roughness according to Andronov - Pontryagin, the foundations of the "topological roughness method" were laid, which allows you to study the roughness and bifurcation of dynamic systems of various nature, in particular synergetic systems, as well as synthesize rough (robustic) control systems [10].

This article presents the main provisions of the "method of topological roughness" developed by the author.

II. Method Bases

In view of the limited scope of this publication, we will focus on some basic provisions of the theory and method of topological roughness, the foundations of which are laid down in the work [9].

Multi-dimensional arrangement considers dynamic system (DS) of n th order

$$\dot{z}(t) = F(z(t)), \quad (1)$$

where $z(t) \in R^n$ is the phase coordinate vector, F is n the dimensional differentiable vector function.

The system (1) is called a topologically rough Andronov-Pontryagin system in some field G of the phase space, if the original system and perturbed system defined in the sub-field \tilde{G} , field G :

$$\tilde{z} = F(\tilde{z}(t)) + f(\tilde{z}(t)), \quad (2)$$

are ε identical in topological sense.

The systems (1) and (2) are ε identical if open fields D, \tilde{D} exist, in the n dimensional phase space such that

$$D \subset \tilde{D} \subset G : \quad \exists \varepsilon, \delta > 0 :$$

$$\text{if } \|f(\tilde{z})\| < \delta, \quad \left| df_i(\tilde{z}) / d\tilde{z}_j \right| < \delta, i, j = \overline{1, n},$$

$$\text{then } \|z\| - \|\tilde{z}\| < \varepsilon,$$

$$\text{or } (\tilde{D}, (2)) \stackrel{\varepsilon}{\equiv} (D, (1)), \quad (3)$$

otherwise, the division of the fields \tilde{D} and D by the tracks of the systems (2) and (1) are ε identical (have the same topological structures with tracks close to ε).

If (3) is not executed, the non rough system (1) on Andronov-Pontryagin.

The topological structure of the phase space of dynamic systems is defined by special tracks and varieties such as special points (equilibrium positions), special lines (separators), closed (periodic) tracks, attracting diversity (attractors).

The work [9] on the basis of the concept of roughness on Andronov-Pontryagin propose the basis of a "method of topological roughness" on the basis of a measure of roughness in the form of a number of conditionality $C \{M\}$ is the matrix of M is the normalized matrix of bringing the system to a canonical diagonal (quasi-diagonal) form at special points of phase space. Here, for the first time introduced the concepts of maximum roughness and minimum non-roughness on the relations of the pair δ and ε .

Definition 1. *Rough in the field of G system (1) is called maximum rough on a set topologically of systems N identical each other if size δ – proximity of systems (1) and (2), bringing to ε – identity, is (for everyone $\varepsilon > 0$) it is maximum.*

Definition 2. Not rough in the field of G system (1) is called minimum non-rough on a set topologically of systems N identical each other if size ε – identity of systems (1) and (2) at which the roughness condition is still satisfied is (for everyone $\delta > 0$) it is minimum.

The condition of reachability of maximum rough and minimum non-rough in the vicinity of special points of phase space is determined by the following theorem proved in the work [5].

Theorem 1. In order for the dynamic system in the vicinity of the hyperbolic special point (z_0) to be maximum rough, and in the vicinity of the non-hyperbolic is minimum non-rough, it is necessary and sufficient to have:

$$M^* = \operatorname{argmin} C\{M\},$$

where M is a matrix of bringing the linear part A of the system (1) at a special point (z_0) to a diagonal (quasi-diagonal) basis, $C\{M\}$ is the condition number of the matrix M .

Theoretical results of the "topological roughness method" obtained in the works [9-11] allow to control the roughness of dynamic systems, the corresponding theorem is proved in these and other works of the author.

The system is considered

$$\dot{z} = Q(z, u), \quad (4)$$

where $z \in R^n$, $u \in R^r$ are, respectively, vectors of phase coordinates and controls of the system, $Q(\bullet)$ is n dimensional nonlinear differentiable vector-function.

The roughness control capabilities are determined by the conditions of the following theorem.

Theorem 2. In order that in a controlled dynamic system (4) described in n -dimensional phase space by linear approximation matrices A , B respectively for phase coordinates and controls, there exists a control $u(t)$ providing in the vicinity of the corresponding special point of the closed system the maximum rough or minimum non-rough, it is necessary and sufficient that the conditions of unbearable permissibility of the Sylvester matrix equation are fulfilled.

Control $u = u(t) \in U$ is sought in the class of feedback systems $u = -Kx$, such that the closed system matrix $F = A - BK$, near special tracks, in particular special points, satisfies the conditions

$$\sigma(F) = \sigma(W),$$

$$MW - AM = -BH, \quad K = HM^{-1},$$

where $W \in R^{n \times n}$ is the diagonal (quasidiagonal) matrix of the state of the canonical model, $H \in R^{m \times n}$ is the matrix set arbitrarily with limitation on the observability of the pair (W, H) , $A \in R^{n \times n}$, $B \in R^{n \times m}$, are the coordinate and control matrices.

Near a special point:

$$F(z) = 0, \quad \dot{z} = Az + Bu,$$

control $u = u(t) \in U$ is synthesized to achieve the required value of $C\{M\}$ using any nonlinear programming methods.

The method of topological roughness also allows to determine bifurcations of dynamic systems based on the criteria developed in the works [9-11]. Moreover, the method presents possibilities of prediction of bifurcations as well as control of bifurcations parameters. The following theorem is proved in the author's doctoral dissertation [12].

Theorem 3. In order for some bifurcation of the topological structure to occur in the field G of the phase space of the multidimensional ($n > 2$) dynamic systems (DS) at the value of the parameter $q = q^*$, $q \in R^p$, it is necessary and sufficient that:

- or 1), in the considered field G , DS there are non-hyperbolic (non-rough) special points (SP), or orbital-unstable limit cycles (LC) for which there is equality

$$C\{M(q^*)\} = \min \sum_{i=1}^p C_i\{M(q)\}, \quad (5)$$

where p is quantity of SP or LC in field G ,

- or 2), in the field G of the DS, there are any rough SP or LC for which the condition is fulfilled

$$C\{M(q^*)\} = \infty. \quad (6)$$

Remark. The type of bifurcation depends, firstly, on which of the conditions (5) or (6) is fulfilled, secondly, on which specific tracks - SP or LC, satisfies these conditions. Thus, for example, chaotic oscillations ("strange attractors") arising from loss of symmetry occur when condition (5) satisfies the SP, and chaotic oscillations occurring through the bifurcation sequences of the doubling period occur when condition (5) meets the LC.

III. Synergetics and Chaos

In modern science, there is increasing interest in its unifying directions, which consider the phenomena of nature and society, living and non-living nature from a single point of view, depending on the properties and characteristics they exhibit. One such area of science is synergetics, which deals with self-organizing processes, phenomena and systems [13, 14]. Synergetics is now invading all fields of science, ranging from natural sciences - physics, chemistry, biology, geology, geophysics, to inaccurate fields of science, such as economics, sociology, psychology, philosophy, image recognition, and in engineering and technology [13], [15].

Many scientists now aim not only to researches synergistic processes and systems, but also to controlling them in order to achieve the desired development and dynamics [16]. One of the phenomena in synergetic systems that are of great interest to researchers in various fields of science is the so-called strange attractors, which represent attractive diversity in phase space with chaotic behavior (chaos) of tracks in these diversity [17]. Research of strange attractors is also interesting because many researchers see the study of this phenomenon as the key to solving the mysteries of the nature of turbulence and chaos in systems of various nature - physical and chemical, to economic and social systems. Moreover, the task of controlling chaos in synergetic systems of different physical nature becomes relevant [13], [16].

The founders of synergetics are rightly outstanding scientists - Belgian chemist and physicist, Nobel laureate Ilya Prigozhin and German physicist Hermann Haken.

In the research and controlling of synergetic systems, the issues of roughness and bifurcation are crucial. One method in studying the properties of roughness and bifurcations of synergetic systems, as well as controlling these properties, is the "topological roughness method," the foundations of which are outlined above.

Further in the work the possibilities of the method are illustrated on example of well-known synergistic systems (circuit) Chua.

IV. Applications of the method to synergetic system

Chua system (circuit) [16].

As is known, the Chua system is an electron circuit with a single nonlinear element that is capable of generating a variety of, particularly chaotic oscillations.

The Chua system is described by the equations:

$$\dot{x} = p(y - f(x)), \dot{y} = x - y + z, \dot{z} = -qy, \quad (7)$$

where $f(x) = M_1x + 0.5(M_1 - M_0)(|x + 1| - |x - 1|)$.

If $p = 9, q = 14.3, M_1 = -6/7, M_0 = 5/7$, in system (7) observed chaotic oscillations.

In this case three special points: $OT_1(0,0,0); OT_{2,3}(\pm 11/6, 0, 11/6)$.

Researches have established that chaotic movements are detected and at values q : $-1.034 < q < -0.49$, and at $q = -3.8$ and $q = 1.05$ maximum roughness of movements in the system (7) is observed, as shown in Fig. 1.

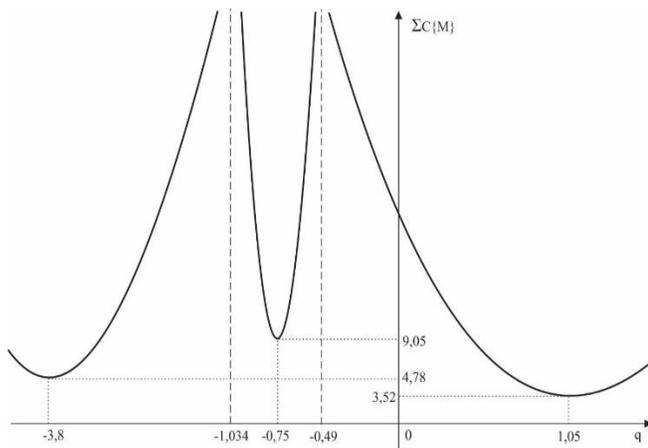


Fig. 1. Dependency of $C \{M\}$ on parameter q in Chua system.

V. Conclusion

The "topological roughness method" discussed in this paper is a method of quantitative research of the roughness and bifurcations of dynamic systems of the broadest class and different physical nature. The possibilities of the method for

research of roughness and bifurcation of systems are shown by example of only one synergetic system, although of course the method has been and can be used to researches a large number of systems, as synergetic systems of various nature - Lorenz, Rossler, Belousov-Zhabotinsky, "predator-prey," Chua, dynamo Rikitake, models of economic systems like Schumpeter and Kaldor, and for research into broader class dynamic systems, in particular in research of oscillatory systems and Hopf bifurcations, Henon map attractor [10] - [12], [15].

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