Minimizing Switchings in Global Bang-Bang Feedback Control

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***Abstract*—Bang-bang feedback controllers can perform a wide range of global control tasks for nonlinear systems, and they are relatively easy to design and implement. This note presents a design framework that yields such controllers with minimal switchings. A low number of switchings reduces controller processing load, reduces mechanical wear, improves performance, and simplifies implementation.**

This note is a didactic and intuitive exposition of some topics from a recent publication by the author.

***Index Terms*—Bang-bang control, Global state feedback, Robust Control, Global stabilization.**

I. Introduction

Bang-bang controllers are among the simplest global controllers. They generate signals that switch between two constant values: the positive and the negative input bounds of the controlled system. It was shown recently that, for a large family of nonlinear systems, bang-bang controllers can approximate performance of any controllers ([1], [2], [3], [4], [5]). Bang-bang controllers are simpler to design and implement than other types of controllers.

The objective of this note is to provide a didactic and intuitive exposition of topics from a recent publication by the author ([6]). The note presents a framework for the design of robust global bang-bang controllers that utilize a minimal number of switchings. Reducing the number of switchings reduces controller processing load, reduces mechanical wear, evens performance, and simplifies implementation. The framework is utilized in the note to design global state-feedback controllers that asymptotically globally stabilize nonlinear systems.

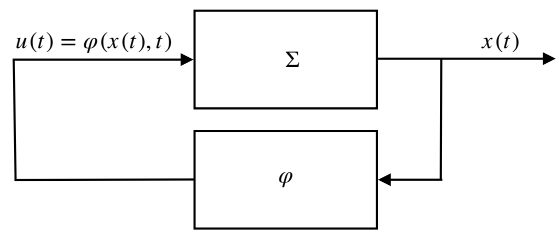


Fig. 1. State feedback diagram

Consider the feedback configuration of Figure 1. Here, the controlled system Σ has its state $x(t)$ as output. The input $u(t)$ of Σ is generated by the state-feedback controller φ. The closed-loop system is denoted by Σ_φ.

The system Σ is subject to modeling uncertainty, and it has an input amplitude bound of $K > 0$. A bang-bang controller φ must be robust to properly control all variants of Σ; the signal $u(t)$ it generates is piecewise constant with components that switch between K and $-K$. The system Σ is also subject to a constraint of $A > 0$ on its state amplitude.

In general, a bang-bang controller can bring the controlled system to the vicinity of the zero state, but it cannot maintain the system at the zero state. To achieve asymptotic stabilization, the control process consists of two stages that depend on the following assumption: (denote by $\rho(\chi) := \{(x^\top x)^{1/2} \leq \chi\}$ the ball of radius χ around the origin of state space.)

Assumption 1. *There is a real number $\chi > 0$ such that the controlled system Σ can be approximated by a stabilizable linear system Λ for all initial states in $\rho(\chi)$.* □

Design Principles 1.
Stage 1: *A robust bang-bang state-feedback controller φ_{bb} drives Σ to $\rho(\chi)$, using a minimal number of switchings.*
Stage 2: *A linear state-feedback controller ϕ asymptotically drives Λ from $\rho(\chi)$ to the origin.* □

This note centers on Stage 1 of Design Principle 1. The construction of linear controllers for Stage 2 is well known (e.g., [7]).

The present note is motivated by the fact that bang-bang controllers are easy to implement and that they – or closely related pseudo bang-bang controllers – can approximate the performance of every controller for large families of nonlinear systems ([1], [4], [5]).

Originally, bang-bang signals were used in minimum-time optimal control of linear systems ([8]). More recent related work can be found in [9], which shows that robustly asymptotically stable closed loops can be achieved by an interpolation technique; and [10], which considers vertex control of linear discrete-time systems. More related papers are referenced in [11], which reviews theoretical aspects of pseudospectral optimal control related to flight control systems. A general background on nonlinear systems can be found in [12], in the references cited there, and in many other publications. An expanded and more technical discussion of the topics covered in this note is found in [6]. As far as we know, there are no other publications that describe general design of bang-bang controllers with minimal switchings.

The note is organized as follows. Notation and basics are in Section II, while Section III examines features of bang-bang controllers. Section IV discusses some facts used in Sections V and VI to derive bang-bang controllers with minimal switchings. Section VII is an example, and Section VIII provides a short summary.

II. Basic notions

Denote by R^q all column vectors with q real components and by R^+ the non-negative real numbers; $|r|$ is the absolute value of a real number r, and $|V| := \max_{i,j} |V_{ij}|$ is the L^∞-norm of a matrix $V = (V_{ij}) \in R^{n\times m}$. The L^∞-norm, also called the *amplitude*, of a function $W(\cdot) : R^+ \to R^{n\times m}$ is $|W|_\infty := \sup_{t\geq 0} |W(t)|$. For a real number $B > 0$, let $[-B, B]^q$ be the set of all $x \in R^q$ with $|x| \leq B$.

A. The class of systems

The controlled systems are time-invariant nonlinear systems of the form

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), t \geq 0, \\ x(0) = x_0; \end{array} \tag{1}$$

here, the time is t, the state is $x(t) \in R^n$, the input is $u(t) \in R^m$, and f is the *recursion function*. Initial states x_0 are in the ball $\rho(\sigma)$, where $\sigma > 0$ is specified.

To incorporate uncertainty about the model of Σ, we represent the recursion function f as a sum of two functions

$$\begin{array}{l} f(x,u) = f_0(x,u) + f_\gamma(x,u), \\ f_0(0,0) = 0; f_\gamma(0,0) = 0. \end{array} \tag{2}$$

Here, f_0 is the specified nominal recursion function, and f_γ is an unknown *uncertainty function*. Both f_0 and f_γ are twice continuously differentiable. The nominal system is

$$\Sigma_0 : \begin{array}{l} \dot{x}(t) = f_0(x(t), u(t)), t \geq 0, \\ x(0) = x_0. \end{array} \tag{3}$$

As Σ has an input amplitude bound of K, permissible input signals are members of the class

$$U(K) := \{u : R^+ \to R^m : |u|_\infty \leq K \text{and } u \text{ is measurable}\}$$

In practice, systems must achieve their control objectives without exceeding safe state amplitudes, as follows.

Assumption 2. *The system Σ can achieve its control objectives without exceeding a state amplitude of $A > 0$.* □

As f_0 and f_γ are twice continuously differentiable, the mean value theorem and continuity of the first derivatives imply the existence of bounds B and γ such that

$$\begin{array}{l} |f_0(x,u) - f_0(x',u')| \leq B(|x - x'| + |u - u'|), \\ |f_\gamma(x,u) - f_\gamma(x',u')| \leq \gamma(|x - x'| + |u - u'|), \end{array} \tag{4}$$

for all $x, x' \in [-A, A]^n$ and $u, u' \in [-K, K]^m$. We call γ the *uncertainty parameter*; it is usually a small number.

The controllers we build are robust – they appropriately control the entire following family of systems.

Notation 1. *Let $K, A, \sigma, \gamma > 0$ be real numbers. Denote by $\mathcal{F}_\gamma(\Sigma_0)$ the family of all systems of the form (1) for which Assumption 2 holds, the recursion function f satisfies (2) and (4), and the following apply.*
(i) *Input signals are bounded by K and state amplitudes are bounded by A.*
(ii) *Initial states are in $\rho(\sigma)$.*
(iii) *All members of $\mathcal{F}_\gamma(\Sigma_0)$ have the same initial state.*
(iv) *All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the same state feedback controller φ.* □

Item *(iii)* is because the state feedback of Figure 1 communicates the initial state. Item *(iv)* indicates robustness: the same controller is appropriate for all of $\mathcal{F}_\gamma(\Sigma_0)$.

By Design Principle 1, no bang-bang controller is needed for initial states in $\rho(\chi)$; there, the system can be asymptotically guided to the origin by a linear feedback controller. As our discussion revolves exclusively around bang-bang controllers, we assume that initial states are in $\rho(\sigma)$ for $\sigma > \chi$.

III. Bang-bang controllers

At each time, a signal produced by a bang-bang controller as input to a system Σ with m inputs is a vector with m entries of K or $-K$, i.e., it is a member of the family

$$\mathcal{C}(m) := \left\{ \upsilon = (\upsilon^1, \upsilon^2, \ldots, \upsilon^m)^\top : |\upsilon^i| = K, i = 1, \ldots, m \right\}$$

Thus, at each time, there are 2^m possible signal values. For instance, for $m = 1$, $\mathcal{C}(1) = \{K, -K\}$; and for $m = 2$, $\mathcal{C}(2) = \{(K,K)^\top, (K,-K)^\top, (-K,K)^\top, (-K,-K)^\top\}$.

Bang-bang controllers generate piecewise constant signals and re-evaluate their value from time to time. The time span between two such re-evaluations is called a *bang-bang step*. The length of bang-bang steps may vary from step to step. Signals may retain the same value for consecutive steps, so the number of switchings may be lower than the number of bang-bang steps. The following comes to limit the time of the control process.

Guideline 1. *The length of a bang-bang step is bounded by $T > 0$.* □

By Design Principle 1, we need to guide the controlled system Σ from an initial state $x_0 \in \rho(\sigma)$ to a state in $\rho(\chi)$. Any state in $\rho(\chi)$ is appropriate; by contrast, x_0 is specific. It is therefore easier to use reverse-time: start from the entire set $\rho(\chi)$, and find a simplest bang-bang signal leading backwards to x_0. Upon reversing time back to original, this yields a simplest bang-bang signal that takes Σ from x_0 to $\rho(\chi)$. This signal is then implemented by feedback.

We start by considering the nominal system Σ_0 of (3). Later, we show that controllers designed for the nominal system can be chosen to be robust to accommodate the entire family $\mathcal{F}_\gamma(\Sigma_0)$.

To reverse the time for Σ_0, substitute $\theta := -t$; this yields the reversed-time system

$$\Gamma_0 : \begin{array}{l} \dot{\xi}(\theta) = -f_0(\xi(\theta), \upsilon(\theta)), \theta \geq 0, \\ \xi(0) = \xi_0, \end{array} \tag{5}$$

with state $\xi(\theta) \in R^n$ and input $\upsilon(\theta) \in R^m$. In order to build flexibility to accommodate all members of $\mathcal{F}_\gamma(\Sigma_0)$, we set the target for Σ_0 as $\rho(\chi/2)$ rather than $\rho(\chi)$. Thus, the reversed-time system Γ_0 starts from $\rho(\chi/2)$.

The derivation of bang-bang controllers depends on certain features of differential equations discussed next.

IV. Boundaries and Sets

During each bang-bang step, the system Γ_0 receives constant input υ, so it is described by a differential equation $\dot{\xi}(t) = -f_0(\xi(t), \upsilon)$, which is, in fact, an autonomous differential equation with no input. As f_0 is twice continuously differentiable, we can use the following fact (e.g., [13]).

Theorem 1. *For any constant input υ and initial state ξ_0, there is a time $\Theta > 0$ for which the differential equation (5) has a unique solution $\xi(t)$, $t \in [0, \Theta]$.* □

In practice, Θ is usually very large or infinite.

The *reachable set* $\mathcal{R}_0(\theta, \upsilon, S)$ is the set of states reached by Γ_0 at time θ from a set $S \subseteq R^n$ of initial conditions, using a constant input $\upsilon \in \mathcal{C}(m)$. The *flow function* $F_0(\theta, \upsilon) : S \to \mathcal{R}_0(\theta, \upsilon, S)$ is defined by

$$\mathcal{R}_0(\theta, \upsilon, S) = F_0(\theta, \upsilon) S. \tag{6}$$

Time invariance of Γ_0 implies the following.

Proposition 1. $F_0(\theta_2, \upsilon)S = F_0(\theta_2 - \theta_1, \upsilon)F_0(\theta_1, \upsilon)S$ *for any times $\theta_1 \leq \theta_2 \leq \Theta$.* □

The flow function has the following important feature (e.g., [13]):

Theorem 2. *The function $F_0(\theta, \upsilon) : S \to \mathcal{R}_0(\theta, \upsilon, S)$ is injective and continuous on S. It is also a continuous function of the time θ.* □

As $F_0(\theta, \upsilon)$ is onto by definition, Theorem 2 yields

Corollary 1. $F_0(\theta, \upsilon) : S \cong \mathcal{R}_0(\theta, \upsilon, S)$ *is a homeomorphism.* □

A. Fully connected sets

The following describes sets with no 'holes'.

Definition 1. *A* fully connected *set $S \subseteq R^n$ is a set that is homeomorphic to a ball in R^n.* □

For instance, ellipsoids are fully connected sets. A fully connected set has no holes; it is completely specified by its outer boundary. Calculating the outer boundary is less computationally taxing than calculating the entire set. For example, if each axis is partitioned into 100 intervals, then calculating the outer boundary requires 100 times fewer computations than calculating all points of the set. The following is a result of the fact that composition of homeomorphisms is a homeomorphism.

Lemma 1. *If $D, D' \subseteq R^n$ are homeomorphic sets and D is fully connected, then so is D'.* □

Next, Corollary 1 and Lemma 1 imply that reachable sets preserve fully connectedness, as follows.

Proposition 2. *If $S \subseteq R^n$ is fully connected, so is $\mathcal{R}_0(\theta, \upsilon, S)$ at all times $\theta \in [0, \Theta]$ and for all inputs $\upsilon \in \mathcal{C}(m)$.* □

In addition, since a homeomorphism maps boundary points to boundary points, we have the following.

Proposition 3. *For any set $S \subseteq R^n$, the boundary of $\mathcal{R}_0(\theta, \upsilon, S)$ is the image of the boundary of S by $F_0(\theta, \upsilon)$.* □

Propositions 3 and 1 imply that boundaries progress recursively in time:

Proposition 4. *The boundary of $\mathcal{R}_0(\theta_2, \upsilon, S)$ is the image of the boundary of $\mathcal{R}_0(\theta_1, \upsilon, S)$ through $F_0(\theta_2 - \theta_1, \upsilon)$, where $\theta_1 \leq \theta_2 \leq \Theta$ and $\upsilon \in \mathcal{C}(m)$.* □

In Sections V and VI, we show that reachable sets are the basis for constructing bang-bang state-feedback controllers. As the initial conditions set for the reversed-time system Γ_0 is $\rho(\chi/2)$ – a fully connected set, all reachable sets are fully connected by Proposition 2; therefore, they are determined by their outer boundaries. By Proposition 4, these boundaries can be computed in a progressive-recursive manner: after computing the boundary at time θ, it can be used to compute the boundary at $\theta + d\theta$, $d\theta > 0$, without calculating the entire time interval from 0 to $\theta + d\theta$.

Recall that a bang-bang step duration is bounded by T. The set $\mathcal{R}_0^*(\upsilon, S)$ of all states accessible by Γ_0 during $[0, T]$ from a set S of initial states, using a constant input $\upsilon \in \mathcal{C}(m)$, is

$$\mathcal{R}_0^*(\upsilon, S) = \bigcup_{\theta \in [0,T]} \mathcal{R}_0(\theta, \upsilon, S). \tag{7}$$

V. Minimal Switchings

Let us examine bang-bang state-feedback controllers for the nominal controlled system Σ_0. We show in Section VI that, if appropriately chosen, such controllers can be robust to properly control the entire family $\mathcal{F}_\gamma(\Sigma_0)$. For the nominal system, we use $\rho(\chi/2)$ as the target set, rather than $\rho(\chi)$, to leave a margin for uncertainties. The next procedure uses the reversed-time system Γ_0 of (5) to build a foundation for controller design.

Procedure 1. *Let $x_0 \in \rho(\sigma)$ be the initial state of Σ_0,* ***and refer to*** *(7)*
Step 0: *Set $S_0 := \rho(\chi/2)$.*
***Step** i**:** For $i = 1, 2, \ldots$, construct recursively the sets*

$$S_i := \bigcup_{\upsilon \in \mathcal{C}(m)} \mathcal{R}_0^*(\upsilon, S_{i-1}), i = 1, 2, \ldots \tag{8}$$

End: *Terminate if there is an integer $k \geq 0$ satisfying*

$$x_0 \in S_k.$$

If k exists, let p be the first such k. □

The computation of the sets $\{S_i\}_{i=0}^p$ of Procedure 1 is not as complex as it might seem. By (8), the set S_i consists of the sets $\mathcal{R}_0^*(\upsilon, S_{i-1})$, which, by (7), consist of the sets $\rho(\chi/2)$, $\mathcal{R}_0(\theta, \upsilon, \rho(\chi/2))$, $\mathcal{R}_0(\theta, \upsilon, \mathcal{R}_0(\theta, \upsilon, \rho(\chi/2)))$, ... As $\rho(\chi/2)$ is fully connected, Proposition 2 shows that these sets are all fully connected, and hence they are determined by their outer boundaries. For the computation of these boundaries, a relatively coarse grid can be used, as discussed in Section VI and demonstrated in the example of Section VII. This results in an acceptable computational effort for most practical systems.

The next construction derives a bang-bang input signal that takes Γ_0 from $\rho(\chi/2)$ to x_0.

Construction 1. *Use the notation of Procedure 1, and assume that $p \geq 1$ (if $p = 0$, then x_0 is in the target set).*
Step 0: *By (8), there is an input $\upsilon_p \in \mathcal{C}(m)$ satisfying $x_0 \in \mathcal{R}_0^*(\upsilon_p, S_{p-1})$. Denote $\xi_p := x_0$.*
***Step** i: For $i = 1, 2, \ldots, p-1$, it follows by (8) that there are a state $\xi_{p-i} \in S_{p-i}$ and an input $\upsilon_{p-i} \in \mathcal{C}(m)$ satisfying $\xi_{p-i} \in \mathcal{R}_0^*(\upsilon_{p-i}, S_{p-i-1})$.*
Therefore, there are a state $\xi_{p-i-1} \in S_{p-i-1}$ and a time $t'_{p-i} \in [0, T]$ for which $\xi_{p-i} = \Gamma_0(\xi_{p-i-1}, \upsilon_{p-i}, t'_{p-i})$.
When $i = p$, then $\xi_0 \in \rho(\chi/2)$. □

In the notation of Construction 1, set

$$t_i := t'_p + t'_{p-1} + \cdots + t'_{p-i}, i = 1, 2, \ldots, p-1,$$

and define the bang-bang signal

$$u_{bb}(t) = \text{switch to } \upsilon_{p-i} \text{ at } t = t_i, i = 1, 2, \ldots, p-1. \quad (9)$$

Let $\Sigma_0(x_0, u, t)$ be the state of Σ_0 at time t, after being driven by input u from initial state x_0. Reversing time to $t = -\theta$, Construction 1 yields

$$\xi_{p-i} = \Sigma_0(x_0, u_{bb}, t_{i-1}), i = 1, 2, \ldots, p, \ t \geq 0. \quad (10)$$

This proves that u_{bb} takes Σ_0 from x_0 to $\rho(\chi/2)$, as follows.

Theorem 3. *Assume that $p \geq 1$ in Procedure 1. The bang-bang input signal u_{bb} of (9) guides Σ_0 from x_0 to a state $\xi_0 \in \rho(\chi/2)$, invoking a minimal number p of bang-bang steps.* □

The states $\xi_0, \xi_1, \ldots, \xi_p$ of Construction 1 determine a bang-bang state-feedback controller, as follows.

Construction 2. *Adopt the notation of Procedure 1 and Construction 1, and assume that p exists.*
If $p = 0$, then $x_0 \in \rho(\chi/2)$ and no bang-bang controller is needed.
If $p > 0$, set $x_i := \xi_{p-i}, i = 0, 1, \ldots, p$, so that (10) yields $x_i = \Sigma_0(x_{i-1}, \upsilon_i, t_i - t_{i-1}), i = 1, 2, \ldots, p$. This leads to the bang-bang state-feedback controller

$$\varphi_{bb}(x) := \text{ switch to } \upsilon_i \text{ at state } x_i, i = 1, \ldots, p-1. \square$$

Our discussion so far proves the following.

Theorem 4. *Assume the notation of Procedure 1 and Construction 2. If there is an integer $p^* \geq 0$ for which $\rho(\sigma) \subseteq S_{p^*}$, then Σ_0 can be taken to $\rho(\chi/2)$ from every initial state $x_0 \in \rho(\sigma)$, using a bang-bang state-feedback controller with no more that $p^* - 1$ switchings.* □

VI. Robust controllers

The controlled system Σ of Figure 1 is an unspecified member of the family $\mathcal{F}_\gamma(\Sigma_0)$. This brings about an uncertainty in the response of the controlled system. This uncertainty must be accommodated by the controller φ. The next statement quantifies the uncertainty. In the statement, $[a]^+$ is the smallest integer bigger than the real number a.

Lemma 2. *Refer to (4). Let u_{bb} be a bang-bang signal with $p \geq 1$ bang-bang steps applied to $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial state x_0. Let x'_k and x_k be the state of Σ and Σ_0, respectively, at the end of step k of u_{bb}. Then, there is an integer $r \geq 1$ such that $|x'_k - x_k| \leq 2^{kr}\gamma A/B$, $k \in \{1, 2, \ldots, p\}$.*
Specifically, $r = [2TB]^+$.

Proof sketch. Let $\varepsilon_k := |x'_k - x_k|$, so that $\varepsilon_0 = |x'_0 - x_0| = |x_0 - x_0| = 0$. Let $t_1 < t_2$ be times during step $k+1$ of u_{bb}. Then, using Guideline 1 and time invariance, we have $t_1, t_2 \in [0, T]$, and, by (4),

$$\begin{aligned}
&|x'(t_2) - x(t_2)| \leq |x'(t_1) - x(t_1)| + \\
&+ \int_{t_1}^{t_2} |f_0(x'(\tau), \upsilon) - f_0(x(\tau), \upsilon)| d\tau \\
&+ \int_{t_1}^{t_2} |f_\gamma(x'(\tau), \upsilon)| d\tau \\
&\leq |x'(t_1) - x(t_1)| \\
&+ B \sup_{t \in [t_1, t_2]} |x'(t) - x(t)|(t_2 - t_1) + \gamma A(t_2 - t_1).
\end{aligned}$$

Select the difference $\eta := t_2 - t_1$ so that $r := T/\eta$ is an integer and $\eta B \leq 1/2$. Then, rearranging terms,

$$\sup_{t \in [t_1, t_2]} |x'(t) - x(t)| \leq 2|x'(t_1) - x(t_1)| + 2\gamma A\eta.$$

Set $t_1 := (i-1)\eta$ and denote $\zeta_i := \sup_{t \in [(i-1)\eta, i\eta]} |x'(t) - x(t)|$ to obtain the recursion

$$\begin{aligned}
&\zeta_{i+1} \leq 2\zeta_i + \gamma A/B, i = 0, 1, \ldots, r-1, \\
&\zeta_0 = \varepsilon_k.
\end{aligned}$$

This yields

$$\zeta_i \leq 2^r(\varepsilon_k + \gamma A/B), i \in \{0, 1, \ldots, r\}.$$

As $\zeta_r = \varepsilon_{k+1}$, we obtain

$$\varepsilon_{k+1} \leq 2^r(\varepsilon_k + \gamma A/B), k = 0, 1, \ldots, p-1, \varepsilon_0 = 0,$$

so that

$$\varepsilon_k \leq 2^{kr}\gamma A/B \leq 2^{pr}\gamma A/B, k \in \{1, 2, \ldots, p\}$$

□

From Lemma 2, we get the following.

Corollary 2. *Let $\delta_1, \delta_2, \ldots, \delta_p > 0$ be real numbers and use the notation of Lemma 2. If $\gamma \leq \min_{k=1,2,\ldots,p} \{\delta_k B/(2^{kr}A)\}$, then $|x'_k - x_k| \leq \delta_k$ for all $k = 1, \ldots, p$.* □

As Σ_0 is guided into $\rho(\chi/2)$, to get every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ into $\rho(\chi)$ we need $\delta_p \leq \chi/2$ in Corollary 2, so that

Corollary 3. *if $\gamma \leq \chi B/(2^{rp+1}A)$, then u_{bb} guides every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ into $\rho(\chi)$.* □

Denote by $\rho(x, \delta)$ a ball of radius $\delta > 0$ centered at x. Then, (6), Corollary 1, (7), and (8) yield

Lemma 3. *Assume that x_1 is chosen to be an interior point of S_{p-1}. Then, there are real numbers $\delta_1, \delta_2, \dots, \delta_p > 0$ for which $\rho(x_i, \delta_i) \subseteq S_{p-i}$, $i = 1, 2, \dots, p$.* □

The next statement is a consequence of Corollaries 2 and 3 and Lemma 3. It lists uncertainty parameters γ that allow proper control of the entire family $\mathcal{F}_\gamma(\Sigma_0)$.

Proposition 5. *Let $\delta_1, \delta_2, \dots, \delta_p$ be as in Lemma 3 and let u_{bb} be a bang-bang input signal that guides Σ_0 from initial state x_0 to $\rho(\chi/2)$. If $\gamma \leq \min_{k=1,2,\dots,p} \{\delta_k B/(2^{kr}A), \chi B/(2^{rp+1}A)\}$, then u_{bb} guides every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ to $\rho(\chi)$, passing $S_1, S_2, \dots, S_{p-1}$ along the way.* □

Proposition 5 allows us to modify Construction 2 to make the state-feedback controller φ_{bb} robust, as follows.

Construction 3. ***Robust bang-bang state-feedback.***

In the notation of Construction 2, Corollaries 2 and 3, Lemma 3, and Proposition 5, the following bang-bang state-feedback controller φ_{bb} takes every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state $x_0 \in \rho(\sigma)$ to $\rho(\chi)$, passing through $S_1, S_2, \dots, S_{p-1}$.

$$\varphi_{bb}(x) = \tag{11}$$

$$\begin{cases} \upsilon_1 & \text{if } x = x_0 \\ \text{switch to } \upsilon_k & \text{when } x \text{ enters } \rho(x_{k-1}, \gamma(2^{(k-1)r}A)/B), \end{cases}$$

$k = 2, 3, \dots, p$. □

The validity of Construction 3 is a consequence of Lemma 2 and Proposition 5. We record this fact as the following central statement of this section.

Theorem 5. *The bang-bang state-feedback controller φ_{bb} of (11) takes every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from a state $x_0 \in \rho(\sigma)$ to $\rho(\chi)$ with a minimal number of bang-bang steps.* □

The controller φ_{bb} of Construction 3 satisfies the requirements of Design Principle 1 Stage 1. It exists whenever Construction 2 terminates in a finite number of steps.

The computational complexity required by Construction 3 is not as overwhelming as might seem at first. It requires calculating the sets S_i, $i = 1, 2, \dots, p$, which are determined by reachable sets $\mathcal{R}_0(\theta, \upsilon, \cdot)$. As we need to find only interior points of the sets S_i, we do not need a highly accurate description of their boundaries. Consequently, it is enough to find $\mathcal{R}_0(\theta, \upsilon, S_{i-1})$ at a few points $\theta \in [0, T]$ in each step. As the latter are unions of fully connected sets, only the outer boundaries of the corresponding fully connected sets need to be calculated. We demonstrate this process in the example of the next section.

VII. Example

Our example is based on the Michaelis-Menten equation, which has important applications in environmental sciences, molecular biology, and pharmacokinetics (e.g., ([14], [15], [16]). The following modified version of the equation describes our controlled system.

$$\Sigma : \quad \begin{aligned} \dot{x}^1(t) &= \frac{a(x^2(t)+2)x^1(t)}{b + x^2(t)} - u(t), \\ \dot{x}^2(t) &= -\frac{cx^2(t)(x^1(t)+2)}{5 + x^2(t)}; \end{aligned}$$

here, the state is $x(t) = (x^1(t), x^2(t))^\top$, and the input is $u(t)$. The constants a, b, and c have nominal values $a_0 = 3$, $b_0 = 5$, and $c_0 = 5$ with uncertainty intervals $a \in [2.9, 3.1]$, $b \in [4.9, 5.1]$, and $c \in [4.9, 5.1]$. The input amplitude bound is $K = 1$; the limit on the duration of each bang-bang step is $T = 3$; initial conditions are in the disk $\rho(0.5)$; the target set is $\rho(0.2)$; and $\mathcal{C}(m) = \mathcal{C}(1) = \{-1, 1\}$. The reversed-time nominal system is

$$\Gamma_0 : \quad \begin{aligned} \dot{\xi}^1(\theta) &= \frac{-3(\xi^2(\theta)+2)\xi^1(\theta)}{5 + \xi^2(\theta)} + u(\theta), \\ \dot{\xi}^2(\theta) &= \frac{5\xi^2(\theta)(\xi^1(\theta)+2)}{5 + \xi^2(\theta)}, \end{aligned}$$

with the state $\xi = (\xi^1, \xi^2)^\top$; the input signal $u(\theta) \in \{-1, 1\}$; and the target set $\rho(\chi/2) = \rho(0.1)$. Proceeding with Procedure 1, calculate the outer boundaries of the reachable sets, starting from the boundary $c(0.1)$ of $\rho(0.1)$. To calculate the set S_1 we need $\mathcal{R}_0^*(-1, c(0.1))$ and $\mathcal{R}_0^*(1, c(0.1))$.

A brief numerical study shows that a good estimate of $\mathcal{R}_0^*(-1, c(0.1))$ can be obtained from the outer boundaries of $\mathcal{R}_0(\theta, -1, c(0.1))$ for $\theta = 0$, 0.833, 1.666, 2.5. The resulting $\mathcal{R}_0^*(-1, c(0.1))$ is given by domain A of Figure 2*(i)*. An approximation of $\mathcal{R}_0^*(1, c(0.1))$ is obtained from the outer boundaries of $\mathcal{R}_0(\theta, 1, c(0.1))$ for $\theta = 0$, 0.5, 1.0, 1.5, as depicted by domain A of Figure 2*(ii)*. These are outcomes of the first reverse bang-bang step.

The second reverse bang-bang step starts from domain A. Exploiting the fact that the reachable sets building domain A are monotone increasing, it is enough to calculate the sets $\mathcal{R}_0^*(1, \mathcal{R}_0(2.5, -1, c(0.1)))$ and $\mathcal{R}_0^*(-1, \mathcal{R}_0(2.5, 1, c(0.1)))$. An approximation of the former can be obtained from the outer boundaries of $\mathcal{R}_0(\theta, 1, \mathcal{R}_0(2.5, -1, c(0.1)))$ for $\theta = 0, 0.366, 0.733, 1.1$; while an approximation of the latter is obtained from $\mathcal{R}_0(\theta, -1, \mathcal{R}_0(1.5, 1, c(0.1)))$ for $\theta = 0$, 0.25, 0.5, 0.75. Then, $\mathcal{R}_0^*(1, \mathcal{R}_0(2.5, -1, c(0.1)))$ is approximately domain B of Figure 2*(i)*, while the domain $\mathcal{R}_0^*(1, \mathcal{R}_0(2.5, -1, c(0.1)))$ is approximately domain B of Figure 2*(ii)*.

It can be seen from Figure 2 that the union of domains A and B includes the entire disk $\rho(0.5)$ of initial states. Therefore, reversing the time to its original direction, we conclude that the target $\rho(0.1)$ is reachable from all initial states in $\rho(0.5)$

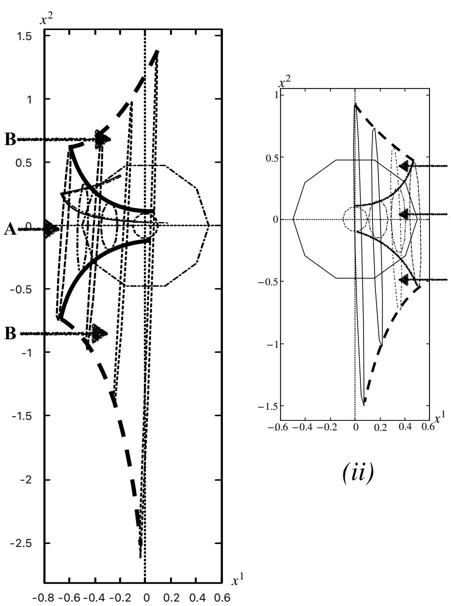
with at most one switching (at most two bang-bang steps). In Figure 2*(i)*, initial states in domain A can reach $\rho(0.1)$ with the input $\upsilon = -1$ (no switching). States in domain B require two bang-bang steps: start with $\upsilon = 1$ and switch to $\upsilon = -1$ at an interior point of domain A.

In Figure 2*(ii)*, initial states in domain A can reach $\rho(0.1)$ with the input $\upsilon = 1$. For initial states in domain B, start with $\upsilon = -1$ and switch to $\upsilon = 1$ at an interior point of domain A.

The following is an example of a bang-bang state-feedback controller that starts from the initial state $x_0 = (-0.2, 0.4)^\top$ in domain B of Figure 2*(i)* (the uncertainty domain of the switching point is given here as a rectangle):

$$\varphi_{bb}(x^1, x^2) = \qquad (12)$$
$$\begin{cases} +1 & \text{if } (x^1, x^2) = (-0.2, 0.4) \text{ (initial state)} \\ \text{switch to } -1 & \text{when } (x^1, x^2) \text{ enters} \\ & [-0.669, -0.665] \times [0.320, 0.322]. \end{cases}$$

The path of Σ_0 with this feedback appears as a thin line in Figure 2*(i)* (a single case is drawn to prevent clutter).



(i) *(ii)*

Fig. 2. Bang-bang step domains

The time response with the feedback controller (12) is shown in Figure 3, together with the signal generated by the feedback controller.

VIII. Conclusion

This note describes the design of robust bang-bang state-feedback controllers that achieve their task with a minimal number of switchings. These controllers help achieve asymptotic stabilization of a large family of nonlinear systems. The design procedure applies to any switching controllers, namely, to any controllers that produce signals with components that switch between two fixed levels.

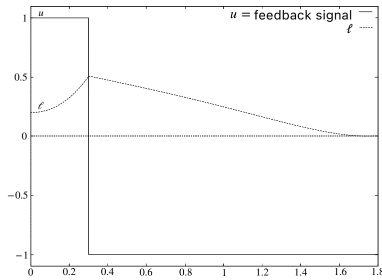


Fig. 3. The time response

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