# Optimal investment in a multi-asset market with borrowing and unbounded random coefficients

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***Abstract*— We consider the problem of optimal investment in a *multi-asset* market consisting of a bond, *two stocks*, possibility of borrowing, unbounded random coefficients, and the power utility from terminal wealth. The resulting optimization problem, due to the higher interest rate for borrowing than for lending, is a *two-input* stochastic optimal control problem with a nonlinear system dynamics and unbounded random coefficients. A certain two-dimensional *piece-wise* completion of squares method and a linear backward stochastic differential equation are used to find an explicit closed-form solution as a linear state-feedback control, the gain of which can have up to *five* different random regimes.**

## I. INTRODUCTION

Consider a multi-asset market of a *bond* with prices $B$ and two stocks with prices $S_i$, $i=1,\ 2$, that are solutions to following equations (for $t\in[0,T]$):

$$\begin{cases} dB(t)=B(t)r(t)dt, \\ dS_i(t)=S_i(t)[\mu_i(t)dt+\sigma_i'(t)dW_1(t)], \quad i=1,2, \\ B(0)>0, \quad S_i(0)>0, \quad i=1,2, \quad \text{are given.} \end{cases} \tag{1}$$

Here $W_1$ is an $m_1$-dimensional standard Brownian motion, and the $m_1$-dimensional volatility vector $\sigma_i$, the bond interest rate $r$, and the rate of return $\mu_i$ are random processes such that equations (1) have unique solutions. Moreover, consider an *investor* with initial wealth $y_0>0$ that holds $v_B(t)$ number of shares in the bond and $v_{S_i}(t)$ number of shares in the stock $i$ at time $t$. The value of investor's portfolio, i. e. the investor's wealth, at time $t$ is thus:

$$y(t):=v_B(t)B(t)+\sum_{i=1}^{2}v_{S_i}(t)S_i(t), \quad t\in[0,T]. \tag{2}$$

This portfolio is called *self-financing* if it has the following dynamics (see, for example, [26], [24]):

$$dy(t)=v_B(t)dB(t)+\sum_{i=1}^{2}v_{S_i}(t)dS_i(t), \quad t\in[0,T]. \tag{3}$$

If we substitute the differentials of $B$ and $S_i$ from (1) into (3), and further knowing that $v_B(t)B(t)=y(t)-\sum_{i=1}^{2}v_{S_i}(t)S_i(t)$, which follows from (2), we obtain (for $t\in[0,T]$):

$$\begin{aligned} dy(t) &= [r(t)y(t)+\sum_{i=1}^{2}(\mu_i(t)-r(t))u_i(t)]dt \\ &+ \sum_{i=1}^{2}u_i(t)\sigma_i'(t)dW_1(t), \end{aligned} \tag{4}$$

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where $u_i(t):=v_{S_i}(t)S_i(t)$ for $t\in[0,T]$. The self-financing portfolio (4) is thus an example of a *linear* stochastic control system with multiplicative noise with the investor's wealth $y$ being the state of the system and the amounts of wealth invested in the stocks $u_i$ being the control variables. The optimal investment problem with expected utility from terminal wealth as the criterion, is the following optimal stochastic control problem:

$$\begin{cases} \max_{u(\cdot)\in\bar{\mathscr{Q}}} \mathbb{E}[U(y(T))], \\ \text{s.t. (4)}, \end{cases} \tag{5}$$

for some suitable admissible set of controls $\bar{\mathscr{Q}}$ and utility function $U$. For a textbook account of problem (5) with a general utility function, see, for example, [26], [24]. The special case with power utility $U(x)=x^\gamma/\gamma$ with $\gamma\in(0,1)$, admits an explicit closed-form solution in a linear state-feedback form (see, for example, [27], for the market with deterministic coefficients). Additionally, for the case of random and possibly *unbounded* coefficients see [22] and [3] for results on the random time horizon, and for other optimality criteria see, for example, [16] for generalised exponential, [19] and [20] for a risk-return criterion, [30] for mean-variance criterion, [7] for a financial benchmark tracking, [21] and [6] for logarithmic utility and Cox-Ingersoll-Ross interest rate model, and the recent papers: [23] for the *robust* portfolio control problem.

A more general and realistic market model than (1) is the one where an investor can *borrow* at a rate $R$ that is higher than the bond rate $r$. In this case one restricts the *borrowed amount* $L(t)$ to be (for $t\in[0,T]$):

$$L(t):=[u'(t)\mathbf{1}-y(t)]^+=\max[0,u'(t)\mathbf{1}-y(t)], \quad t\in[0,T],$$

where $u:=[u_1,\ u_2]'$ and $\mathbf{1}:=[1,\ 1]'$, as is not reasonable to borrow with the higher rate $R$ and at the same time invest in the bond with a lower rate $r$. The equation of investor's wealth (4) now becomes a *nonlinear* stochastic control system with multiplicative noise as follows (for $t\in[0,T]$):

$$\begin{aligned} &dy(t)=[r(t)y(t)+u'(t)(\mu(t)-r(t))]dt-(R(t)- \\ &\times r(t))L(t)dt+u'(t)\sigma(t)dW_1(t)=[r(t)y(t)+u'(t)(\mu(t) \\ &-r(t))]dt-(R(t)-r(t))[u'(t)\mathbf{1}-y(t)]^+dt \\ &+u'(t)\sigma(t)dW_1(t), \end{aligned} \tag{6}$$

where

$$\mu(t):=[\mu_1(t),\ \mu_2(t)]', \quad \sigma(t):=[\sigma_1(t),\ \sigma_2(t)]'.$$

In the market with a single stock, the optimal investment and *consumption* problem with borrowing at the higher rate $R$ was first considered in [18]. There it is assumed that the coefficients are *constant* and an explicit solution was found in the case of a power utility from consumption in *infinite* horizon. Moreover, a more general case of a market with random coefficients was considered in [15], where an *existence* result was obtained for a general utility function. The explicit solution to the optimal investment problem with the *logarithmic utility* was obtained in the setting of random coefficients only, whereas for the power utility it was assumed that the coefficients are deterministic coefficients. In [15], in addition to leaving open the problem of finding an explicit closed-form solution in a market with random coefficients and power utility, the existence result given there does not apply to some typical interest rate models. These are the ones given as solutions to stochastic differential equations (see, for example, [31], [13], [14], [28]), which in general are not bounded (a lower bound on $r$ is assumed in [15]). Moreover, in [15] it was assumed that the coefficients are adapted to the filtration of $W_1$ and thus only complete markets are considered. This does not permit for the source of randomness in the interest rates to be different from that of stock (see, for example, [17], [29]). In the recent series papers [8], [9], [10], [11], [12], [1], [2], [3] several cases of the optimal investment problem in an *incomplete* market with borrowing, random interest rates, and the power utility have been considered, and it was succeeded in obtaining an explicit closed-form solution in the following cases: in [8] the interest rate is assumed to be quadratic-affine with independent source of uncertainty as compared to the stock; in [11] this was generalised further to permit for a general class of such coefficients; in [2] this was generalised to allow for a general class of such coefficients along with random time horizon instead of fixed horizon; in [10] the Hull-White model for the interest rate was used which has the same source of uncertainty as the stock; in [3] the Heston volatility model was used which has the same source of uncertainty as the stock; in [9] the market with a Markovian switching coefficients is considered; whereas in [12] and [1] markets with certain *combined* random interest rate models are introduced; [4] for a quadratic-affine interest rates and Heston stochastic volatility is considered. A common assumption in all of these papers on borrowing, and similarly to [27], is that an investor is only permitted to invest in a single stock.

In this paper, we study the optimal investment problem in an incomplete market with random and possibly unbounded coefficients, with higher interest rate for borrowing than lending, with wealth equation (6), and the following power utility from terminal wealth:

$$J(u(\cdot)) := -\frac{1}{\gamma}\mathbb{E}\left[y^{\gamma}(T))\right], \quad \gamma \in (0,1), \tag{7}$$

which, due to its minus sign, is to be minimized. Due to borrowing, market incompleteness, multi-assets market, and possibly unbounded coefficients, this problem is not covered by the results of [15], and is a generalisation to the two-stock market of the results in [11], [4]. In order to solve this problem, we apply the theory of backward stochastic differential equations (BSDEs) and a certain two-dimensional piece-wise completion of square method. The unique explicit closed-form solution to this problem is obtained as a linear state-feedback control, the gain of which can have up to *five* different regimes instead of three as in [11], [4]. In §II we give the precise formulation of the problem to be solved, whereas in §III we give its explicit solution. Note that in order to simplify the presentation, in what follows we have omitted the argument $t$ whenever convenient.

## II. Problem formulation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which an $m_1$-dimensional standard Brownian motion $(W_1(t),\ t \geq 0)$ and an $m_2$-dimensional standard Brownian motion $(W_2(t),\ t \geq 0)$ are defined and assumed to be independent. We further consider the filtrations $(\mathscr{F}(t),\ t \in [0,T])$ and $(\mathscr{F}_2(t),\ t \in [0,T])$, where $\mathscr{F}(t)$ is the augmentation of $\sigma\{W_1(s), W_2(s)\colon 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$, and $\mathscr{F}_2(t)$ is the augmentation of $\sigma\{W_2(s)\colon 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathscr{F}$. Our assumptions on the market coefficients are as follows: the coefficients $r$, $R$, $\mu$, and $\sigma$, are assumed to be $\mathscr{F}$-adapted stochastic processes that are possibly unbounded; the stock volatility matrix $\sigma$ is assumed to satisfy the following positivity property $\sigma(t)\sigma'(t) > 0$ *a.e.* $t \in [0,T]$ *a.s.*, which ensures the absence of *arbitrage* (see, for example, [13]). By defining the following processes (for $t \in [0,T]$):

$$\begin{aligned} a(t) &:= [a_1,\ a_2]' := [\mu_1(t) - r(t),\ \mu_2(t) - r(t)]', \\ b(t) &:= R(t) - r(t), \end{aligned}$$

the wealth equation (6) becomes

$$\begin{cases} dy = [ry + u'a - b[u'\mathbf{1} - y]^+]\,dt + u'\sigma dW_1, \quad t \in [0,T], \\ y(0) = y_0 > 0. \end{cases} \tag{8}$$

Note that our market is *incomplete* in general since the number of uncertainty sources (which in this case is $m_1 + m_2$) is larger than the number of stocks (which in this case is 2). The cost functional that we consider is the power utility from terminal wealth as given by (7), which we repeat here for convenience:

$$J(u(\cdot)) = -\frac{1}{\gamma}\mathbb{E}\left[y^{\gamma}(T))\right], \quad \gamma \in (0,1).$$

The corresponding optimal investment problem to be considered is the following optimal stochastic control problem:

$$\begin{cases} \min\limits_{u(\cdot) \in \mathscr{A}} J(u(\cdot)), \\ \text{s.t. } (8), \end{cases} \tag{9}$$

where $\mathscr{A}$ is a suitable set of admissible controls to be defined next. In order to give the definition of $\mathscr{A}$, and later in §III to state the solution to problem (9), we define the following (for $t \in [0,T]$):

$$\alpha_0 := [0,1]', \tag{10}$$

$$\alpha_1 := [1,-1]', \tag{11}$$

$$\xi(t) := (1-\gamma)\sigma(t)\sigma'(t), \quad \eta(t) := \alpha_1'\xi(t)\alpha_1.$$

For simplicity, we define the matrix $\xi$ and its inverse in the following form (for $t\in[0,T]$):

$$\xi \quad := \quad \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix},\ \xi^{-1} := \frac{1}{\xi_1\xi_4-\xi_2\xi_3}\begin{bmatrix} \xi_4 & -\xi_2 \\ -\xi_3 & \xi_1 \end{bmatrix},$$

Further, let $f := \xi_1\xi_4-\xi_2\xi_3, \quad \forall\ t\in[0,T]$, and denote the elements of $\xi^{-1}$ as:

$$\begin{aligned}\xi_1^{-1} &:= f^{-1}\xi_4, \quad \xi_2^{-1} := -f^{-1}\xi_2, \\ \xi_3^{-1} &:= -f^{-1}\xi_3, \quad \xi_4^{-1} := f^{-1}\xi_1.\end{aligned}$$

Moreover, we introduce the following processes (for $t\in[0,T]$):

$$\begin{aligned} g_1 &:= -\frac{1}{2}a'\xi^{-1}a, \quad g_2 := -\frac{1}{2}(a-b\mathbf{1})'\xi^{-1}(a-b\mathbf{1})-b, \\ g_3 &:= \frac{1}{2}\Big\{-\frac{1}{2\eta}\left(\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)\right)^2+\alpha_0'(\xi\alpha_0-2a)\Big\}, \end{aligned}$$

$$\begin{aligned} h_1 &:= (\xi^{-1}a)'\mathbf{1}, \quad h_2 := (\xi^{-1}a-\xi^{-1}b)'\mathbf{1}, \\ k_1 &:= \xi^{-1}a, \quad k_2 := \xi^{-1}(a-b\mathbf{1}), \\ k_3 &:= \left[-\frac{\alpha_1'\xi(\alpha_0-\xi^{-1}a)}{\eta}, 1+\frac{\alpha_1'\xi(\alpha_0-\xi^{-1}a)}{\eta}\right]', \end{aligned}$$

$$\Psi(t) := \begin{cases} g_1 & \text{if } h_1\le 1 \text{ and } h_2\le 1, \\ g_1 & \text{if } h_1\le 1 \text{ and } h_2>1, \\ g_2 & \text{if } h_1\le 1 \text{ and } h_2>1, \\ g_2 & \text{if } h_1>1 \text{ and } h_2>1, \\ g_3 & \text{if } h_1>1 \text{ and } h_2\le 1, \end{cases}$$

$$\Gamma(t) := \begin{cases} k_1 & \text{if } h_1\le 1 \text{ and } h_2\le 1, \\ k_1 & \text{if } g_1\le g_2,\ \ h_1\le 1, \text{ and } h_2>1, \\ k_2 & \text{if } g_1>g_2,\ h_1\le 1, \text{ and } h_2>1, \\ k_2 & \text{if } h_1>1, \text{ and } h_2>1, \\ k_3 & \text{if } h_1>1 \text{ and } h_2\le 1, \end{cases}$$

$N(\nu) := \nu'\xi\nu+\nu'a+b[\nu'\mathbf{1}-1]^+$, where $\nu(t) := u(t)/y(t)$.

*Theorem 2.1:* The minimisation problem

$$\min_{\nu\in\mathbb{R}^2} N(\nu)$$

has a unique solution. The minimizer is $\nu^* := \Gamma$, and the corresponding minimum is $N(\nu^*) = \Psi$.

*Proof.* Let $N(\nu)$ be formulated using the completion of square method as follows:

$$\begin{aligned} N(\nu) &= \nu'\xi\nu+\nu'a+b[\nu'\mathbf{1}-1]^+ \\ &= \frac{1}{2}\left[\left(\nu-\xi^{-1}a\right)'\xi\left(\nu-\xi^{-1}a\right)-a'\xi^{-1}a\right]I_{(\nu'\mathbf{1}\le 1)} \\ &+ \frac{1}{2}\Big\{\left[\nu-\xi^{-1}(a-b\mathbf{1})\right]'\xi\left[\nu-\xi^{-1}(a-b\mathbf{1})\right] \\ &- (a-b\mathbf{1})'\xi^{-1}(a-b\mathbf{1})-2b\Big\}I_{(\nu'\mathbf{1}>1)}. \end{aligned} \tag{12}$$

where $I_{(\cdot)}$ is the indicator function. The minimizer of $N(\nu)$ and its corresponding minimum value are given by the following five cases.

Case (1): if $h_1\le 1$ and $h_2\le 1$, then $\nu^*=k_1$ and $N(\nu^*)=g_1$.

Case (2): if $g_1\le g_2,\ h_1\le 1$, and $h_2>1$, then $\nu^*=k_1$ and $N(\nu^*)=g_1$.

Case (3): if $g_1>g_2,\ h_1\le 1$, and $h_2>1$, then $\nu^*=k_2$ and $N(\nu^*)=g_2$.

Case (4): if $h_1>1$ and $h_2>1$, then $\nu^*=k_2$ and $N(\nu^*)=g_2$.

Case (5): if $h_1>1$ and $h_2\le 1$, then $\nu^*=k_3$ and $N(\nu^*)=g_3$.

The derivation of the minimizer of $N(\nu)$ and its minimum value for the cases (1)-(4) above is more straightforward and follows directly from the given conditions and restrictions on the interval. Our aim now is to demonstrate $\nu^*$ and $N(\nu^*)$ for case (5) above. This corresponds to the case of the vector $\nu=[\nu_1\quad \nu_2]'$ that satisfies the condition:

$$\nu_1+\nu_2=1. \tag{13}$$

From (13) it follows that $\nu_2=1-\nu_1$, and further that $\nu=\alpha_0+\alpha_1\nu_1$, where $\alpha_0$ and $\alpha_1$ are defined in (10) and (11), respectively. The function $N(\nu)$ can now be written as:

$$\begin{aligned} N(\nu) &= \frac{1}{2}\left[\left(\nu-\xi^{-1}a\right)'\xi\left(\nu-\xi^{-1}a\right)-a'\xi^{-1}a\right]I_{(\nu'\mathbf{1}\le 1)} \\ &= \frac{1}{2}\left[\left(\alpha_1\nu_1+\alpha_0-\xi^{-1}a\right)'\xi\left(\alpha_1\nu_1+\alpha_0-\xi^{-1}a\right)-a'\xi^{-1}a\right] \\ &= \frac{1}{2}\Big[\alpha_1'\xi\alpha_1\nu_1^2+2\nu_1\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)+\left(\alpha_0-\xi^{-1}a\right)'\xi \\ &\times\left(\alpha_0-\xi^{-1}a\right)-a'\xi^{-1}a\Big] \\ &= \frac{1}{2}\Big\{\alpha_1'\xi\alpha_1\Big[\nu_1^2+2\frac{\nu_1\alpha_1'\xi}{\alpha_1'\xi\alpha_1}\left(\alpha_0-\xi^{-1}a\right) \\ &+\left(\frac{\alpha_1'\xi(\alpha_0-\xi^{-1}a)}{\alpha_1'\xi\alpha_1}\right)^2\Big] \\ &-\frac{[\alpha_1'\xi(\alpha_0-\xi^{-1}a)]^2}{2\alpha_1'\xi\alpha_1}+\alpha_0'\left(\xi\alpha_0-2a\right)\Big\} \\ &= \frac{1}{2}\Big\{\eta\left[\nu_1+\frac{\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)}{\eta}\right]^2 \\ &-\frac{1}{2\eta}\left[\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)\right]^2+\alpha_0'(\xi\alpha_0-2a)\Big\}. \end{aligned}$$

We conclude that $\nu_1^* := -\frac{\alpha_1'\xi(\alpha_0-\xi^{-1}a)}{\eta}$, and

$$\nu_2^* = 1-\nu_1^* = 1+\frac{\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)}{\eta}.$$

Therefore, the minimizer of $N(\nu)$ is:

$$\nu^* = \left[-\frac{\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)}{\eta}, 1+\frac{\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)}{\eta}\right]'. \quad (14)$$

Clearly, the corresponding minimum value of $N(\nu)$ is:

$$\begin{aligned} N(\nu^*) &= \frac{1}{2}\Bigg\{-\frac{1}{2\eta}\left[\alpha_1'\xi\left(\alpha_0-\xi^{-1}a\right)\right]^2 \\ &+ \alpha_0'(\xi\alpha_0-2a)\Bigg\} = g_3. \qquad \square \end{aligned}$$

We further introduce the following BSDE:

$$\begin{cases} dP = \Big[\gamma P(\Psi-r)\Big]dt + z'dW_2, \quad t\in[0,T], \\ P(T) = -1/\gamma \quad a.s.. \end{cases} \quad (15)$$

Note that this is a linear BSDE, with possibly unbounded coefficients. To prove that there exist a unique solution pair $(P,z)$ to the BSDE (15), we begin by deriving some sufficient conditions which place certain requirements on the adaptivity and integrability of given coefficients. These conditions are as follows:
$(i)$ the process $(\Psi(t)-r(t),\ t\in[0,T])$ is $\mathscr{F}_2$-adapted;
$(ii)$ $\mathbb{E}\Big[\phi^2(T)\Big] < \infty$.
where $\phi$ is defined as (for $t\in[0,T]$):

$$\phi(t) := \exp\left\{-\gamma\int_0^t(\Psi(s)-r(s))\,ds\right\}.$$

*Lemma 2.1:* If conditions $(i)$ and $(ii)$ hold, then there exists a unique solution pair $(P,z)$ to (15), and $P$ is given explicitly as:

$$P(t) = \frac{-1}{\gamma}\mathbb{E}\left[e^{\int_t^T -\gamma(\Psi-r)ds}\Big|\mathscr{F}_2(t)\right]. \quad (16)$$

The following integrability further holds:

$$\mathbb{E}\left[\int_0^T \phi^2(t)z'(t)z(t)dt\right] < \infty. \quad (17)$$

*Proof.* The process $\phi$ is the unique solution to the following differential equation:

$$\begin{cases} d\phi = -\gamma\phi(\Psi-r)dt, \quad t\in[0,T], \\ \phi(0)=1. \end{cases} \quad (18)$$

By Itô's product rule, the differential of $\phi P$ is given as:

$$\begin{aligned} d(\phi P) &= \phi dP + (d\phi)P + (d\phi)dP \\ &= \phi P\gamma(\Psi-r)dt + \phi z'dW_2 - \phi\gamma(\Psi-r)Pdt \\ &= \phi z'dW_2. \end{aligned} \quad (19)$$

By integrating both sides of the above equation (19) from $t$ to $T$, we obtain:

$$\frac{-\phi(T)}{\gamma} - \phi(t)P(t) = \int_t^T \phi(s)z'(s)dW_2(s). \quad (20)$$

As $\phi^2(T)$ is a square integrable random variable (see condition $(ii)$), by martingale representation there exists a process $z$ such that (20) holds for all $t\in[0,T]$ and the process $\phi z$ is square integrable. Next, to derive an explicit formula of $P$, we take the conditional expectation of both sides of equation (20), we obtain:

$$\begin{aligned} P(t) &= \frac{-1}{\gamma}\mathbb{E}\left[\frac{\phi(T)}{\phi(t)}\Big|\mathscr{F}_2(t)\right] \\ &= \frac{-1}{\gamma}\mathbb{E}\left[\frac{e^{-\int_0^T\gamma(\Psi-r)ds}}{e^{-\int_0^t\gamma(\Psi-r)ds}}\Big|\mathscr{F}_2(t)\right] \\ &= \frac{-1}{\gamma}\mathbb{E}\left[e^{\int_t^T-\gamma(\Psi-r)ds}\Big|\mathscr{F}_2(t)\right]. \end{aligned}$$

$\square$

The admissible set of controls $\mathscr{A}$ is defined as the set of all $\mathscr{F}$-adapted scalar processes $u$ under which the wealth equation (8) has a unique and strictly positive strong solution, i.e. $y(t)>0$ $a.s.$ for all $t\in[0,T]$, and the processes (for $t\in[0,T]$):

$$G_1(t) := \int_0^t y^\gamma(s)z'(s)dW_2(s), \quad (21)$$

$$G_2(t) := \int_0^t P(s)y^\gamma(s)\gamma(s)\frac{u'(s)}{y(s)}\sigma(s)dW_1(s), \quad (22)$$

are $\mathscr{F}(t)$-martingales. It is clear that strict positivity on the wealth $y$ is required in order to avoid investor's bankruptcy, whereas the requirements (21) and (22) guarantee that the expected values of two stochastic integrals appearing in the solution to optimal investment problem (9) are zero (see the proof of Theorem 3.1 in §III). This completes the formulation of the problem to be solved.

## III. Solution to the Optimal Investment Problem

In order to state the solution to problem (9), we introduce the following two additional conditions (for $t\in[0,T]$):
$(\lambda i)$ $\mathbb{E}\left[\sup_{t\in[0,T]}\frac{Y^{2\gamma}(t)}{\phi^2(t)}\right] < \infty$,
$(\lambda ii)$ $\mathbb{E}\left[\int_0^T Y^{2\gamma}(t)P^2(t)\Gamma^2(t)\sigma'(t)\sigma(t)dt\right] < \infty$,
Here the process $Y$ is defined as (for $t\in[0,T]$):

$$\begin{aligned} Y(t) &:= y_0\exp\Bigg\{\int_0^t\Big[\Pi(s)-(\Lambda(s)\Lambda'(s))/2\Big]ds \\ &+ \int_0^t\Lambda'(s)dW_1(s)\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} \Pi(t) &:= r(t)+\Gamma'(t)a(t)-b(t)[\Gamma'(t)\mathbf{1}-1]^+, \\ \Lambda(t) &:= \Gamma'(t)\sigma(t). \end{aligned}$$

*Theorem 3.1:* Let the conditions $(i)$, $(ii)$, $(\lambda i)$, and $(\lambda ii)$ hold. There exists a unique solution $u^*$ to the optimal investment problem (9) given by:

$$u^*(t) = \Gamma(t)y(t), \quad t\in[0,T].$$

The optimal cost functional is $J(u^*(\cdot)) = -P(0)y_0^\gamma$.
*Proof.* Due to conditions $(i)$ and $(ii)$, by Lemma 2.1 there exists a unique solution pair to the BSDE (15). By Itô's

formula, the differential of $y^\gamma$ is:

$$dy^\gamma = \gamma y^{\gamma-1}\left[ry + u'a - b[u'\mathbf{1} - y]^+\right]dt + \frac{1}{2}\gamma(\gamma-1)y^{\gamma-2}$$
$$\times u'\sigma\sigma' u dt + \gamma y^{\gamma-1}u'\sigma dW_1$$
$$dy^\gamma = \gamma y^{\gamma}\left[r + \nu'a - b[\nu'\mathbf{1} - y]^+\right]dt + \frac{1}{2}\gamma(\gamma-1)y^{\gamma}$$
$$\times \nu'\sigma\sigma'\nu dt + \gamma y^\gamma \nu'\sigma dW_1$$

By Itô's product rule, the differential of $Py^\gamma$ is:

$$\begin{aligned}
d(Py^\gamma) &= y^\gamma dP + P(dy^\gamma) + (dP)dy^\gamma \\
&= y^\gamma[(-P\gamma r + \gamma P\Psi)]dt + z'y^\gamma dW_2 + P\gamma y^\gamma\Big(r + \nu'a \\
&\quad - b[\nu'\mathbf{1} - 1]^+ + \frac{1}{2}(\gamma-1)\nu'\sigma\sigma'\nu\Big)dt + P\gamma y^\gamma \nu'\sigma dW_1 \\
&= y^\gamma\Big[-P\gamma r + \gamma P\Psi + r\gamma P + \gamma P\nu'a - \gamma Pb\left[\nu'\mathbf{1} - 1\right]^+ \\
&\quad + \frac{\gamma}{2}(\gamma-1)P\nu'\sigma\sigma'\nu\Big]dt + P\gamma y^\gamma\nu'\sigma dW_1 + y^\gamma z' dW_2
\end{aligned}$$

Integrating both sides of the above equation from 0 to $T$ gives:

$$\begin{aligned}
P(T)y^\gamma(T) &= P(0)y_0^\gamma + \int_0^T y^\gamma\left[\gamma P\Psi + \gamma P\nu'a - \gamma P\right. \\
&\quad \left.\times b\left[\nu'\mathbf{1} - 1\right]^+ + \frac{\gamma}{2}(\gamma-1)P\nu'\sigma\sigma'\nu\right]dt + \int_0^T P\gamma y^\gamma \nu'\sigma dW_1 \\
&\quad + \int_0^T y^\gamma z' dW_2. \qquad (23)
\end{aligned}$$

By taking the expectation of both sides of (23), we obtain the following for any admissible control (note that due to the integrability requirements (21) and (22), the expectation of two stochastic integrals are zero):

$$\begin{aligned}
\frac{-1}{\gamma}\mathbb{E}[y^\gamma(T)] &= P_0y_0^\gamma + \mathbb{E}\Big\{\int_0^T y^\gamma\Big(\gamma P\Psi + \gamma P\nu'a \\
&\quad - \gamma Pb\left[\nu'\mathbf{1} - 1\right]^+ + \frac{\gamma}{2}(\gamma-1)P\nu'\sigma\sigma'\nu\Big)dt\Big\}.
\end{aligned}$$

where $P_0 = P(0)$. Therefore, we obtain the following integral representation of the cost functional $J$ for all admissible controls:

$$\begin{aligned}
J(u(\cdot)) &= P_0y_0^\gamma + \mathbb{E}\Big\{\int_0^T y^\gamma\Big[\gamma P\Psi + \gamma P\Big(\nu'a - b\left[\nu'\mathbf{1} - 1\right]^+ \\
&\quad + \frac{1}{2}(\gamma-1)\nu'\sigma\sigma'\nu\Big)\Big]dt\Big\} \\
&= P_0y_0^\gamma + \mathbb{E}\left[\int_0^T \gamma P(\Psi - N(\nu))dt\right]. \qquad (24)
\end{aligned}$$

We can thus obtain the following lower bound for the cost functional (24) for any admissible control:

$$J(u(\cdot)) = P_0y_0^\gamma - \mathbb{E}\left[\int_0^T \gamma P(N(\nu) - \Psi)dt\right] \geq P_0y_0^\gamma.$$

This lower bound is achieved if and only if $\nu(t) = \nu^*(t) := \Gamma(t)$ *a.e.* $t \in [0,T]$ *a.s.*, or equivalently, if and only if $u^*(t) = \Gamma(t)y(t)$ *a.e.* $t \in [0,T]$ *a.s.*. The corresponding optimal cost functional is $J(u^*(\cdot)) = P_0y_0^\gamma$. Thus, if $u^* \in \mathscr{A}$, then $u^*$ is the unique solution to the optimal investment problem (9). We now show that indeed $u^*(\cdot) \in \mathscr{A}$. By substituting $u^*$ in the wealth equation (8) we obtain:

$$\begin{cases} dy = \Pi y dt + \Lambda' y dW_1, & t \in [0,T], \\ y(0) = y_0 > 0. \end{cases}$$

As this is a linear stochastic differential equation, it has a unique strong solution (see [22]). As $y(t) = Y(t)$, it is clear that $Y(t) > 0$ *a.s.* for all $t \in [0,T]$. In order to show the integrability requirements (21) and (22) hold under the optimal control $u^*$, thus by using the Burkholder-Davis-Gundy inequality (see, for example, Theorem 1.5.4 of [32]) we conclude that there exists a constant $Q$ such that:

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{t\in[0,T]}\left|\int_0^t y^\gamma(s)z'(s)dW_2(s)\right|\right\} \\
&\leqslant Q\mathbb{E}\left\{\int_0^T \frac{y^{2\gamma}(s)}{\phi^2(s)}\phi^2(s)z'(s)z(s)ds\right\}^{\frac{1}{2}} \\
&\leqslant Q\mathbb{E}\left\{\sup_{t\in[0,T]}\left[\frac{y^{2\gamma}(t)}{\phi^2(t)}\right]\int_0^T \phi^2(s)z'(s)z(s)ds\right\}^{\frac{1}{2}} \\
&\leqslant \frac{Q}{2}\mathbb{E}\left\{\sup_{t\in[0,T]}\left[\frac{y^{2\gamma}(t)}{\phi^2(t)}\right] + \int_0^T \phi^2(s)z'(s)z(s)ds\right\} < \infty,
\end{aligned}$$

which is due to condition $(\lambda i)$, and the fact that process $\phi z$ is square integrable. It now follows from Corollary 7.22 of [25] that the process $G_1$ is a martingale. The process $G_2$ under the control $u^*$ becomes:

$$G_2(t) := \int_0^t P(s)y^\gamma(s)\Gamma'(s)\sigma(s)dW_1(s),\ t \in [0,T].$$

As a consequence of condition $(\lambda ii)$, this stochastic integral has a square-integrable integrand, and it is thus a martingale. We conclude that $u^*(\cdot) \in \mathscr{A}$ and it is the unique solution to the optimal investment problem (9). □

The optimal control $u^*$ is thus in a linear state-feedback form with gain $\Gamma$. As can be noted from its definition, the process $\Gamma$, depending on the values of market coefficients, can take up to five different values.

## IV. CONCLUSIONS

We have considered an optimal investment problem in a two-stock market with borrowing, random coefficients (which are not assumed to be bounded), with different interest rates for borrowing and lending, and a power utility from terminal wealth. We have found an explicit closed-form solution to this problem by developing a certain two-dimensional piece-wise completion of squares method. This shows a rare case of a two input nonlinear stochastic optimal control problem that permits a fully explicit solution. In this context, a number of additional interesting problems can be considered. For instance, incorporating consumption in the investor's wealth equation would significantly increase the problem's difficulty, as would the consideration of other sources of market randomness, such as the case with Markovian switching coefficients (as in [9]) or more general interest rate models (as in [12], [1]).

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