Robust Remote Estimation of Lipschitz-type Nonlinear Retarded State-multiplicative Systems

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***Abstract*— We consider the remote estimation of retarded, Lipschitz-type non-linear, discrete-time systems with state-multiplicative stochastic uncertainties in their state-space model and deterministic uncertain parameters that lie in a given polytope. We address the problems of robust uncertain vertex-dependant remote H_∞ estimation that are based on time-delayed measurements of these systems. The solution to the estimation problem is achieved by solving a set of linear matrix inequalities. The theory is demonstrated by a numerical example that compares the various methods in our theory.**

Keywords : remote estimation, stochastic systems, Lipschitz nonlinearity, robust filtering

I. INTRODUCTION

We address the problems of robust H_∞ and H_2 estimations of measurement-delayed, discrete-time, state-multiplicative Lipschitz-type non-linear systems. Following a series of matrix inequalities transformations, we derive robust filters that are based on a set of tractable Linear Matrix Inequalities (LMIs) condition with a minimal number of tuning parameters.

Modeling nonlinear systems to be of a Lipschitz-type has been proven to be very fruitful in many engineering fields. Indeed, there is a large volume of literature concerning the solution of various control and observer-dependant problems for both: deterministic systems (see [2] - [6]) and various stochastic systems ([7] - [9]). In practice, the field of robotic control and estimation has been a major platform for modeling the various typical nonlinearities found in robotic arms and vehicles as Lipschitz-type ones (see for example [2] for a study of one link flexible model).

Recently [6] , the problem of H_∞ robust estimation of **deterministic delay-free** Lipschitz non-linear discrete-time systems has been solved for both: polytopic and deterministic norm-bounded uncertain systems via the application of a non-linear filter, where a LMI condition is used to extract the filter parameters. The work of [6] deals with deterministic systems and it does not account for measurement delay, albeit dealing with deterministic systems.

The stochastic case, where the robust estimation has been applied to state multiplicative retarded systems, has been treated in [10]. In this latter work the delay does not appear in the measurement process thus limiting the work to cases where the estimation is based on an on-line measurement of a combination of the system states. It does not apply to cases where a remote control loop is applied to a system where decision making policies are made based on the evolution of the system dynamics in a considerably remote place such that the measurement signals delay affects the estimation process.

Similar cases where a remote estimation is applied can be abundantly found in networked control systems for various reasons (see [11] - [13]). For example in [11],the problem of remote estimation is explored in cases where due to bandwidth and energy constrains, sensor scheduling is applied and where the information (receiving multiple packets) is obtained via a wireless networks.

The H_∞ control and estimation theory of **linear systems** with stochastic uncertainties has been developed over the last 40 years (see [14] and the references therein) where numerous types of the latter problems have been extensively studied in various settings [i.e. delay-free [14]- [18] and delayed [19]-[27] systems, switched systems [28], biological systems [28] etc.). The robust estimation problem of retarded linear systems has been solved in [29], [30]. Comparing to the linear setting, the study of the Lipschitz-type nonlinear counterpart of the latter problems, has been mainly centered around delay-free systems (see, for example [3] - [5]) and to a lesser extent to stochastic state-multiplicative retarded systems, mainly in the continuous-time setting [7] - [9]. To the best of our knowledge, no solution can be found, as yet, in the literature that solved the Lipschitz-type nonlinear estimation problem of retarded state-multiplicative discrete-time systems.

In this paper, we consider the estimation problem of uncertain stochastic Lipschitz-type nonlinear retarded systems. In our system we allow for a time-varying delay in the measurement matrix where the system nonlinearities appear in both the "deterministic" part and the "stochastic" part of the system states, and where the uncertain stochastic parameters multiply, for simplicity, only the non-delayed states of the system. We first solve the robust H_∞ problem following the solution of the estimation problem for nominal systems. In the robust case, we apply a modified version of the Finsler lemma which allows us to obtain a vertex-dependent solution. This solution has a significant advantage over the "quadratic" approach where a single Lyapunov function is assigned to the entire uncertain polytope of the system. The quadratic solution is, in fact, a special case of the vertex-dependent one and thus it attains, in most cases, a lower performance compared to the more general case.

The paper is organized as follows: Following the prob-

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lem formulation in Section 2, we bring in Section 3.1 the solution to the remote estimation problem for nominal systems followed by an improved robust vertex-dependant H_∞ estimation for the uncertain polytopic case, in Section 3.2. In Section 4, we bring an example that compares the various solution methods for the estimation of stochastic nonlinear retarded systems.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega,\mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k\in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$\begin{array}{c}||\{f_k\}||^2_{\tilde{l}_2}=E\{\sum_0^\infty ||f_k||^2\}=\sum_0^\infty E\{||f_k||^2\}<\infty,\\ \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n),\end{array}\tag{1}$$

where $||\cdot||$ is the standard Euclidean norm and E is the expectation operator. We also denote by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of [17] (see Definition 2.1, p. 927, there).

II. Problem Formulation

We consider the following nonlinear discrete-time system:

$$\begin{array}{l}x_{k+1}=f(x_k)x_k+A_1x_{k-h}+D(x_k)\nu_k+B_1w_k\\ x_l=0,\ l\leq 0,\ y_k=C_2x_{k-h}+D_{21}n_k\end{array}\tag{2a,c}$$

with the objective vector

$$z_k=C_1x_k,\tag{3}$$

where the time delay is denoted by h,

$$\begin{array}{l}f(x_k)=A_0+H_1F(x_k)E_1 \text{ and } D(x_k)=\\ (D+H_2F(x_k)E_2)x_k,\end{array}\tag{4a,b}$$

and where $x_k\in\mathcal{R}^n$ is the system state vector, $w_k\in\mathcal{R}^q$ is the exogenous disturbance signal, $n_k\in\mathcal{R}^p$ is the measurement noise signal, $y_k\in\mathcal{R}^m$ is the measured output and $z_k\in\mathcal{R}^r$ is the objective function signal. The nonlinear part $F(x)$, which is in $\mathcal{R}^{q_1\times q_2}$, is assumed to be bounded. Namely,

$$F(x_k)^TF(x_k)\leq I,\quad \forall x_k\in R^n.\tag{5}$$

The variable $\{\nu_k\}$ is a zero-mean real scalar white-noise sequence that satisfies:

$$E\{\nu_k\nu_j\}=\delta_{kj},\ \forall k,j\quad \geq 0,$$

and the matrices in (2a,c), (3) are constant matrices of appropriate dimensions.

We treat the following problem :

i) H_∞ estimation:

We first consider the nominal system of (2a-c) and (3) and the estimator of the form:

$$\begin{array}{l}\hat{x}_{k+1}=A_c\hat{x}_k+B_cy_k,\\ \hat{z}_k=C_c\hat{x}_k.\end{array}\tag{6}$$

We denote

$$e_k=x_k-\hat{x}_k,\quad \text{and}\quad \bar{z}_k=z_k-\hat{z}_k,\tag{7}$$

and we consider the following cost function:

$$J_F\triangleq ||\bar{z}_k||^2_{\tilde{l}_2}-\gamma^2[||w_k||^2_{\tilde{l}_2}+||n_{k+1}||^2_{\tilde{l}_2}].\tag{8}$$

Given $\gamma>0$, we first seek an estimate $C_c\hat{x}_k$ of C_1x_k over the infinite time horizon $[0,\infty)$ such that J_F of (8) is negative for all nonzero w_k and n_k where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$ and $n_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty];\mathcal{R}^p)$.

Once conditions are derived for the existence of a solution to the above nominal filtering problem, we assume that the system parameters are uncertain but known to reside in the following polytope:

$$\bar{\Omega}\triangleq\left[\begin{array}{ccccccc}A_0 & A_1 & B_1 & C_1 & C_2 & D_{21} & D\end{array}\right],\tag{9}$$

which is described by the vertices:

$$\bar{\Omega}=\mathcal{C}o\{\bar{\Omega}_1,\bar{\Omega}_2,...,\bar{\Omega}_N\},\tag{10}$$

where

$$\bar{\Omega}_i\triangleq\left[\begin{array}{ccccccc}A_0^{(i)} & A_1^{(i)} & B_1^{(i)} & C_1^{(i)} & C_2^{(i)} & D_{21}^{(i)} & D^{(i)}\end{array}\right]\tag{11}$$

and where N is the number of vertices. In other words:

$$\bar{\Omega}=\sum_{i=1}^N\bar{\Omega}_if_i,\quad \sum_{i=1}^N f_i=1,\quad f_i\geq 0.\tag{12}$$

Similarly to the solution for the nominal system, we seek, given $\gamma>0$, an estimate $C_c\hat{x}_k$ of C_1x_k over the infinite time horizon $[0,\infty)$ such that J_F of (8) is negative over the uncertainty polytope, for all nonzero w_k and n_k, where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$ and $n_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty];\mathcal{R}^p)$.

III. Remote H_∞ Estimation

A. The nominal-case

We consider the system of (2a-c) and (3) and the general type filter of (6). Denoting $\xi_k^T\triangleq[x_k^T\quad \hat{x}_k^T]$ and $\bar{w}_k^T\triangleq[w_k^T\ \ n_k^T]$ we obtain the following augmented system:

$$\begin{array}{l}\xi_{k+1}=[\tilde{A}_0+\tilde{H}_1\tilde{F}(x_k)\tilde{E}_1]\xi_k+\tilde{B}\bar{w}_k+\tilde{A}_1\xi_{k-h}\\ +[\tilde{D}+\tilde{H}_2\tilde{F}(x_k)\tilde{E}_2]\xi_k\nu_k,\ \tilde{z}_k=\tilde{C}\xi_k,\quad \xi_l=0,\ l\leq 0,\end{array}\tag{13}$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & A_c \end{bmatrix},\ \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ B_c C_2 & 0 \end{bmatrix},$$
$$\tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ 0 & B_c D_{21} \end{bmatrix},\ \tilde{C}^T = \begin{bmatrix} C_1^T \\ -C_c^T \end{bmatrix},$$
$$\tilde{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{H}_1 = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (14a\text{-}i)$$
$$\tilde{H}_2 = \begin{bmatrix} H_2 & 0 \\ 0 & 0 \end{bmatrix},\quad \tilde{E}_1 = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\text{and} \quad \tilde{E}_2 = \begin{bmatrix} E_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying a recent result of [30], the following Bounded Real Lemma (BRL) condition is obtained for the augmented system of (13) and (14) to be stable and to attain H_∞ -norm less than γ:

Lemma 1: Consider the system of (13) and (14) and (4). The system is exponentially stable in the mean-square sense and, for a prescribed scalar $\gamma > 0$ and a given scalar tuning parameter $\epsilon_f > 0$, the requirement of $\|\bar{z}_k\|^2_{\tilde{l}_2} < \gamma^2 \|\bar{w}_k\|^2_{\tilde{l}_2}$ is achieved for all $k > 0$ and $\bar{w}_k \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^{q+r})$, if there exist $n \times n$ matrices $\tilde{P} > 0$, $\tilde{R}_1 > 0$, and $\tilde{P}_M$ and positive scalars $\bar{\rho}_1$ and $\bar{\rho}_2$ that satisfy $\hat{\Gamma} < 0$ where $\hat{\Gamma}$ is the following LMI:

$$\begin{bmatrix}
\hat{\Gamma}_{11} & \hat{\Gamma}_{12} & 0 & 0 & \hat{\Gamma}_{15} & 0 & \tilde{C}_1^T & \hat{\Gamma}_{18} & \bar{\rho}_1 \tilde{E}_1^T & 0 & \bar{\rho}_3 \tilde{E}_2^T & 0 \\
* & -\tilde{P} & \hat{\Gamma}_{23} & P_M & 0 & \tilde{P}\tilde{B}_1 & 0 & 0 & 0 & \tilde{P}\tilde{H}_1 & 0 & 0 \\
* & * & -\tilde{R}_1 & 0 & \hat{\Gamma}_{35} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_f \tilde{P} & \hat{\Gamma}_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_f \tilde{P} & \hat{\Gamma}_{56} & 0 & 0 & 0 & \bar{\Gamma}_{5,10} & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \hat{\Gamma}_{88} & 0 & 0 & 0 & \bar{\Gamma}_{8,12} \\
* & * & * & * & * & * & * & * & -\bar{\rho}_1 I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\bar{\rho}_1 I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\bar{\rho}_2 I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\bar{\rho}_2 I
\end{bmatrix}, \qquad (15)$$

where

$\hat{\Gamma}_{11} = -\tilde{P} + \tilde{R}_1,\ \hat{\Gamma}_{12} = \tilde{A}_0^T \tilde{P} + \tilde{P}_M^T,$
$\hat{\Gamma}_{15} = h\epsilon_f[\tilde{A}_0^T \tilde{P} + \tilde{P}_M^T] - \epsilon_f \tilde{P} h,\ \hat{\Gamma}_{18} = \tilde{D}^T(\tilde{P} + h^2 \epsilon_f \tilde{P}),$
$\hat{\Gamma}_{23} = \tilde{P}\tilde{A}_1 - \tilde{P}_M,\ \hat{\Gamma}_{35} = h\epsilon_f[\tilde{A}_1^T \tilde{P} - \tilde{P}_M^T],$
$\hat{\Gamma}_{45} = -h\epsilon_f \tilde{P}_M^T,\ \hat{\Gamma}_{56} = h\epsilon_f \tilde{P}\tilde{B}_1,\ \bar{\Gamma}_{5,10} = h\epsilon_f \tilde{P}\tilde{H}_1,$
$\hat{\Gamma}_{8,8} = -(\tilde{P} + h^2 \epsilon_f \tilde{P}),$ and $\bar{\Gamma}_{8,12} = (\tilde{P} + h^2 \epsilon_f \tilde{P})\tilde{H}_2.$

Defining $\tilde{Q} = \tilde{P}^{-1}$ and denoting the following partitions:

$$\tilde{Q} = \begin{bmatrix} X & M^T \\ M & T \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} Y & N^T \\ N & W \end{bmatrix},\ J = \begin{bmatrix} X^{-1} & Y \\ 0 & N \end{bmatrix}, \qquad (16a\text{-}c)$$

we multiply (15) by $\hat{J} = diag\{\tilde{Q}J, \tilde{Q}J, \tilde{Q}J, \tilde{Q}J, \tilde{Q}J, I, I, \tilde{Q}J, I, I, I, I\}$, from the right, and by $\hat{J}^T$, from the left.

Denoting also $\bar{R}_q = J^T \tilde{Q}\tilde{R}_1 \tilde{Q} J$, $\bar{X} = X^{-1}$, $\bar{X}_y = \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix}$, and $\tilde{Q}_M = J^T \tilde{Q}\tilde{P}_M \tilde{Q} J$, the following condition is obtained:

$$\begin{bmatrix}
\hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & 0 & 0 & \hat{\Upsilon}_{15} & 0 & J^T \tilde{Q}\tilde{C}^T & \hat{\Upsilon}_{18} \\
* & -\bar{X}_y & \hat{\Upsilon}_{23} & \tilde{Q}_M & 0 & J^T \tilde{B}_1 & 0 & 0 \\
* & * & -\bar{R}_q & 0 & \hat{\Upsilon}_{35} & 0 & 0 & 0 \\
* & * & * & -\epsilon_f J_{\tilde{Q}} & \hat{\Upsilon}_{45} & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_f J_{\tilde{Q}} & h\epsilon_f J^T \tilde{B}_1 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 \\
* & * & * & * & * & * & * & -J_{\tilde{Q}} \\
* & * & * & * & * & * & * & *
\end{bmatrix}$$
$$+\bar{\Upsilon}^T \Phi \bar{\Upsilon} < 0, \qquad (17)$$

$$\bar{\Upsilon} = \begin{bmatrix}
\bar{\rho}_1 \hat{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{\Upsilon}^T_{2,10} & 0 & 0 & \hat{\Upsilon}^T_{5,10} & 0 & 0 & 0 \\
\bar{\rho}_2 \hat{E}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{\Upsilon}^T_{8,12}
\end{bmatrix}$$

$\Phi = diag\{\frac{1}{\bar{\rho}_1} I,\ \frac{1}{\bar{\rho}_1} I,\ \frac{1}{\bar{\rho}_2} I,\ \frac{1}{\bar{\rho}_2} I\}$ and where,

$J_{\tilde{Q}} = J^T \tilde{Q} J, \quad \bar{\epsilon}^2 = 1 + \epsilon_f h^2, \quad \hat{\Upsilon}_{11} = -\bar{X}_y + \bar{R}_q,$
$\hat{\Upsilon}_{12} = J^T \tilde{Q}\tilde{A}_0^T J + \tilde{Q}_M,$
$\hat{\Upsilon}_{15} = \epsilon_f h[J^T \tilde{Q}\tilde{A}_0^T J + \tilde{Q}_M] - \epsilon_f h J^T \tilde{Q} J,$
$\hat{\Upsilon}_{18} = \bar{\epsilon} J^T \tilde{Q}\tilde{D}^T J, \quad \hat{\Upsilon}_{23} = J^T \tilde{A}_1 \tilde{Q} J - \tilde{Q}_M,$
$\hat{\Upsilon}_{2,10} = \begin{bmatrix} \bar{X}H_1 & 0 \\ YH_1 & 0 \end{bmatrix}, \quad \hat{\Upsilon}_{35} = \epsilon_f h \hat{\Upsilon}_{23}^T,$
$\hat{\Upsilon}_{45} = -h\epsilon_f \tilde{Q}_M,$
$\hat{\Upsilon}_{5,10} = h\epsilon_f \begin{bmatrix} \bar{X}H_1 & 0 \\ YH_1 & 0 \end{bmatrix}, \quad \hat{\Upsilon}_{8,12} = \bar{\epsilon}^2 \begin{bmatrix} \bar{X}H_3 & 0 \\ YH_3 & 0 \end{bmatrix},$
$\hat{E}_1 = \begin{bmatrix} E_1 & E_1 \\ 0 & 0 \end{bmatrix},\ \hat{E}_2 = \begin{bmatrix} E_2 & E_2 \\ 0 & 0 \end{bmatrix}.$

Denoting $\Delta = Y - \bar{X}$, choosing $N = I$, and carrying out the various multiplications the following LMI condition is obtained:

$$\begin{bmatrix}
\hat{\Psi}_{11} & \hat{\Psi}_{12} & 0 & 0 & \hat{\Psi}_{15} & 0 & J^T \tilde{Q}\tilde{C}^T & \hat{\Psi}_{18} \\
* & -\bar{X}_y & \hat{\Psi}_{23} & \tilde{Q}_M & 0 & \hat{\Psi}_{26} & 0 & 0 \\
* & * & -\bar{R}_q & 0 & \hat{\Psi}_{35} & 0 & 0 & 0 \\
* & * & * & -\epsilon_f \bar{X}_y & \hat{\Psi}_{45} & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_f \bar{X}_y & h\epsilon_f J^T \tilde{B}_1 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 \\
* & * & * & * & * & * & * & -J_{\tilde{Q}}
\end{bmatrix}$$
$$+\tilde{\Psi}^T \bar{\Phi} \tilde{\Psi} < 0, \qquad (18)$$

where

$$\tilde{\Psi} = \begin{bmatrix} \bar{\rho}_1\hat{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{\Psi}^T_{2,10} & 0 & 0 & \hat{\Psi}^T_{5,10} & 0 & 0 & 0 \\ \bar{\rho}_2\hat{E}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{\Psi}^T_{8,12} \end{bmatrix},$$

and where

$$\bar{\Phi} = diag\{\frac{1}{\bar{\rho}_1}I,\ \frac{1}{\bar{\rho}_1}I,\ \frac{1}{\bar{\rho}_2}I,\ \frac{1}{\bar{\rho}_2}I\}.$$

and,

$$\hat{\Psi}_{11} = -\bar{X}_y + \bar{R}_q,$$

$$\hat{\Psi}_{12} = \begin{bmatrix} A_0^T\bar{X} & A_0^TY - \Delta A_c^T \\ A_0^T\bar{X} & A_0^TY \end{bmatrix} + \tilde{Q}_M,$$

$$\hat{\Psi}_{15} = \epsilon_f h\hat{\Psi}_{12} - \epsilon_f h \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix},$$

$$\hat{\Psi}_{17} = \begin{bmatrix} C_1^T + \Delta C_c^T \\ C_1^T \end{bmatrix}, \quad \hat{\Psi}_{18} = \bar{\epsilon}\begin{bmatrix} D^T\bar{X} & D^TY \\ D^T\bar{X} & D^TY \end{bmatrix},$$

$$\hat{\Psi}_{23} = \begin{bmatrix} \bar{X}A_1 & \bar{X}A_1 \\ YA_1 + B_cC_2 & YA_1 + B_cC_2 \end{bmatrix} - \tilde{Q}_M,$$

$$\hat{\Psi}_{26} = \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & B_cD_{21} \end{bmatrix}, \ \hat{\Psi}_{2,10} = \begin{bmatrix} \bar{X}H_1 & 0 \\ YH_1 & 0 \end{bmatrix},$$

$$\hat{\Psi}_{35} = \epsilon_f h\hat{\Psi}^T_{23},$$

$$\hat{\Psi}_{56} = h\epsilon_f \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & B_cD_{21} \end{bmatrix}, \quad \hat{\Psi}_{5,10} = h\epsilon_f\hat{\Psi}_{2,10},,$$

$$\hat{\Psi}_{8,12} = \begin{bmatrix} \bar{X}H_2 & 0 \\ YH_2 & 0 \end{bmatrix}, \quad \text{and} \quad \bar{\epsilon}^2 = 1 + \epsilon_f h^2. \tag{19a-n}$$

We thus arrive at the following result:

Theorem 1 Consider the system of (2a-c), (3), and (4). For a prescribed scalar $\gamma > 0$ and a positive tuning scalar ϵ_f, there exists a filter of the structure (6) that achieves $J_F < 0$, where J_F is given in (8), for all nonzero $w_k \in \tilde{l}^2([0,\infty);\mathcal{R}^q)$ and $n_k \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$, if there exist: $n \times n$ matrices $\bar{X} > 0$, $Y > 0$, $2n \times 2n$ matrix $\bar{R}_q > 0$, $n \times n$ matrix A_c , $2n \times 2n$ matrix $\tilde{Q}_M$, a $n \times m$ matrix B_c, a $r \times n$ matrix C_c, and scalars $\bar{\rho}_1$ and $\bar{\rho}_2$ that satisfy (19).

B. The uncertain case

In the uncertain case, a robust filter is obtained either by applying the quadratic solution which is based on the BRL condition of Theorem 1 (where a single Lyapunov function is assigned to all the uncertainty interval) or by adopting the following vertex dependent approach:
Starting with (19), we define

$$\Gamma = diag\{\bar{E},\bar{E},\bar{E},\bar{E},\bar{E},I,I,\bar{E},I,I,I,I\}, \text{ and } \bar{E} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}. \tag{20}$$

We then multiply (19) by Γ and Γ^T, from the left and the right, respectively. Denoting the dependence on the i-th vertex of the uncertainty polytope by a superscript i we obtain the following requirement for qll $i = 1, ..., N$.

$$\Phi^i \stackrel{\Delta}{=} \begin{bmatrix} \bar{\psi}^i_{11} & \bar{\psi}^i_{12} & 0 & 0 & \bar{\psi}^i_{15} & 0 & \bar{\psi}^i_{17} & \bar{\psi}_{18} \\ * & -V^i & \bar{\psi}^i_{23} & \hat{P}^i_M & 0 & \bar{\psi}^i_{26} & 0 & 0 \\ * & * & -\hat{R}^i_p & 0 & h\epsilon_f\bar{\psi}^{i,T}_{23} & 0 & 0 & 0 \\ * & * & * & -\epsilon_f V^i & -h\epsilon_f\hat{P}^i_M & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_f V^i & \bar{\psi}^i_{56} & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 \\ * & * & * & * & * & * & * & -V^i \end{bmatrix} + \tilde{\psi}^{T,i}\tilde{\Phi}\tilde{\psi}^i < 0, \tag{21}$$

where

$$\tilde{\psi}^i = \begin{bmatrix} \bar{\rho}_1\hat{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\psi}^{T,i}_{2,10} & 0 & 0 & \bar{\psi}^{T,i}_{5,10} & 0 & 0 & 0 \\ \bar{\rho}_2\hat{E}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\psi}^{T,i}_{8,12} \end{bmatrix},$$

and where

$$\tilde{\Phi} = diag\{\frac{1}{\bar{\rho}_1}I,\ \frac{1}{\bar{\rho}_1}I,\ \frac{1}{\bar{\rho}_2}I,\ \frac{1}{\bar{\rho}_2}I\}.$$

where

$$V^i = \begin{bmatrix} \Delta^i & -\Delta^i \\ -\Delta^i & Y^i \end{bmatrix}, \ \Delta^i = Y^i - \bar{X}^i, \ \hat{R}^i_q = \bar{E}\bar{R}^i_q\bar{E}^T,$$

$$\bar{\psi}^i_{11} = -V^i + \hat{R}^i_q, \quad \bar{\psi}^i_{12} = \begin{bmatrix} \Delta^i A_c^T & -\Delta^i A_c^T \\ -A_0^{i,T}\Delta^i & A_0^{i,T}Y^i \end{bmatrix} + \hat{P}^i_M,$$

$$\bar{\psi}^i_{15} = h\epsilon_f(\bar{\psi}^i_{12} - V^i), \quad \bar{\psi}_{17} = \begin{bmatrix} \Delta^i C_c^T \\ C_1^{i,T} \end{bmatrix},$$

$$\bar{\psi}^i_{18} = \bar{\epsilon}\begin{bmatrix} 0 & 0 \\ -D^{i,T}\Delta^i & D^{i,T}Y^i \end{bmatrix},$$

$$\bar{\psi}^i_{23} = \begin{bmatrix} 0 & -\Delta^i A^i_1 - B_cC^i_2 \\ 0 & Y^iA^i_1 + B_cC^i_2 \end{bmatrix} - \hat{P}^i_M,$$

$$\bar{\psi}^i_{26} = \begin{bmatrix} -\Delta^i B^i_1 & -B_cD^i_{21} \\ Y^iB^i_1 & B_cD^i_{21} \end{bmatrix}, \quad \bar{\psi}_{2,10} = \begin{bmatrix} -\Delta H_1 & 0 \\ YH_1 & 0 \end{bmatrix},$$

$$\bar{\psi}^i_{56} = h\epsilon_f\begin{bmatrix} -\Delta^i B^i_1 & -UD^i_{21} \\ Y^iB^i_1 & UD^i_{21} \end{bmatrix}, \quad \bar{\psi}^i_{5,10} = h\epsilon_f\bar{\psi}_{2,10},$$

$$\bar{\psi}^i_{8,12} = \begin{bmatrix} -\Delta H_3 & 0 \\ YH_3 & 0 \end{bmatrix}, \quad \text{and} \quad Q^i_M = E\bar{Q}^i_M E^T.$$

We denote: $\Phi^i_1 = \begin{bmatrix} \tilde{\Phi}^i_1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\text{where } \tilde{\Phi}^i_1 \stackrel{\Delta}{=} \begin{bmatrix} 0 & \bar{\Phi}^i_{12} & 0 & 0 & \bar{\Phi}^i_{15} & 0 & \bar{\Phi}^i_{17} & 0 \\ \bar{\Phi}^i_{21} & 0 & \bar{\Phi}^i_{23} & 0 & 0 & \bar{\Phi}^i_{26} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{\Phi}^i_{51} & 0 & \bar{\Phi}^i_{53} & 0 & 0 & \bar{\Phi}^i_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{\Phi}^i_{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S^i = diag\{\Delta^i,\ Y^i\},$$
$$\hat{\Delta}^i = diag\{\Delta^i, I_n, S^i, I_{4n}, S^i, I_{r+p+q}, S^i, S^i, 0\}, \tag{22a-c}$$

and

$$\bar{\Phi}^i = \Phi^i - \hat{\Delta}^i\Phi_1^i - \Phi_1^{iT}\hat{\Delta}^{iT},\ \bar{\Phi}_{12}^i = \begin{bmatrix} I_n \\ 0 \end{bmatrix}\begin{bmatrix} A_c^T & -A_c^T \end{bmatrix},$$

$$\bar{\Phi}_{15}^i = h\epsilon_f \begin{bmatrix} I_n \\ 0 \end{bmatrix}\begin{bmatrix} A_c^T & -A_c^T \end{bmatrix},\ \bar{\Phi}_{17}^i = \begin{bmatrix} I_n \\ 0 \end{bmatrix} C_c^T,$$

$$\bar{\Phi}_{21}^i = \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} 0 & A_0^i \end{bmatrix},\ \bar{\Phi}_{23}^i = \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} 0 & A_1^i \end{bmatrix},$$

$$\bar{\Phi}_{26}^i = \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} B_1^i & 0 \end{bmatrix},$$

$$\bar{\Phi}_{51}^i = \epsilon_f h \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} 0 & A_0^i \end{bmatrix} \quad \bar{\Phi}_{53}^i = \epsilon_f h \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} 0 & A_1^i \end{bmatrix},$$

$$\bar{\Phi}_{56}^i = \epsilon_f h \begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} B_1^i & 0 \end{bmatrix},\ \bar{\Phi}_{81}^i = \bar{\epsilon}\begin{bmatrix} -I_n \\ I_n \end{bmatrix}\begin{bmatrix} 0 & D^i \end{bmatrix}. \tag{23}$$

We note that although $\bar{\Phi}^i$ includes the matrices Y^i and Δ^i (via V^i), these matrices do not multiply there the system matrices $A^i,\ B_1^i,\ D^i$ and the filter matrix B_c.

To find the required filter, we seek $S^i,\ i = 1,...N,\ A_c$, B_c and C_c that solve the following set of LMIs:

$$\bar{\Phi}^i + \hat{\Delta}^i\Phi_1^i + \Phi_1^{iT}\hat{\Delta}^{iT} < 0,\ \ i = 1,...,N. \tag{24}$$

The latter set of inequalities is, however, not convex in the decision variables S^i. We therefore apply the Finsler lemma and consider the following set of LMIs:

$$\begin{bmatrix} \bar{\Phi}^i + G\Phi_1^i + \Phi_1^{iT}G^T & \hat{\Delta}^i - G + \Phi_1^{iT}\bar{H} \\ * & -\bar{H} - \bar{H}^T \end{bmatrix} < 0,\ \ i = 1,...,N, \tag{25}$$

where G and $\bar{H} \in \mathcal{R}^{v\ xv}$.

In order to guarantee positive definite X^i and Y^i we add to the LMIs in (25) the following two requirements.

$$\begin{bmatrix} 0 & I_n \end{bmatrix} S^i \begin{bmatrix} 0 \\ I_n \end{bmatrix} > 0 \qquad \text{and} \tag{26}$$

$$\begin{bmatrix} 0 & I_n \end{bmatrix} S^i \begin{bmatrix} 0 \\ I_n \end{bmatrix} - \begin{bmatrix} I_n & 0 \end{bmatrix} S^i \begin{bmatrix} I_n \\ 0 \end{bmatrix} > 0,\ \ i = 1,...N.$$

Remark 1: To reduce the computational burden one may choose in the above

$$\bar{H} = diag\{\tilde{\epsilon}\bar{G}_1,\ \epsilon_1 I_n,\ \bar{H}_1, \epsilon_2 I_{4n}, \bar{H}_2, \epsilon_3 I_{r+p+q}, \bar{H}_3, \bar{H}_4, 0\},$$

$$G = diag\{\bar{G}_1,\ I_n,\ \bar{G}, I_{4n}, \bar{G}, I_{r+p+q}, \bar{G}, \bar{G}, 0\}, \tag{27a,b}$$

where $\bar{G} = diag\{\bar{G}_1,\ \bar{G}_2\}$, $\bar{G}_1$ and $\bar{G}_2 \in \mathcal{R}^{nxn}$, where $\epsilon_1,\ \epsilon_2$ and ϵ_3 are decision scalars and $\tilde{\epsilon}$ is a tuning scalar. Denoting $\bar{G}_{A_c} = \bar{G}_1 A_c^T$ and $\bar{G}_{C_c} = \bar{G}_1 C_c^T$, we thus obtain the following result:

Theorem 2 Consider the system of (2a-c) and (3), (4) where the system matrices lie within the polytope $\bar{\Omega}$ of (9). For a prescribed scalar $\gamma > 0$, and positive tuning scalars ϵ_f and $\tilde{\epsilon}$, there exists a filter of the structure (6) that achieves $J_F < 0$, where J_F is given in (8), for all nonzero $w \in \tilde{l}^2([0,\infty);\mathcal{R}^q),\ n \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$, if there exist: $2n \times 2n$ matrices $\bar{R}_q > 0,\ \tilde{Q}_M$, and $S^i > 0,\ i = 1,...,N,\ n \times n$ matrices $\bar{G}_1,\ \bar{G}_2, \bar{H}_1,\ \bar{H}_2,\ \bar{H}_3,\ \bar{H}_4$ and $\bar{G}_{A_c}$, a $n \times m$ matrix B_c, a $n \times r$ matrix $\bar{G}_{C_c}$ and decision scalars $\epsilon_1,\ \epsilon_2,\ \epsilon_3$ that satisfy (25) - (27). In this case the filter parameters are

$$A_c = \bar{G}_{A_c}^T \bar{G}_1^{-T}, \quad \text{and} \quad C_c = \bar{G}_{C_c}^T \bar{G}_1^{-T}. \tag{28a-c}$$

IV. Example - Robust Filtering

We consider the system of (2a-c) and (3) with the following system matrices:

$$A = \begin{bmatrix} 0.1 & 0.6 \pm a \\ -1 & -0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0.252 \\ 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.225 \\ 0.45 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.5 & 04 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

where $E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $H_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0.01 \end{bmatrix}$, $H_2 = H_3 = E_2 = E_3 = 0$.

Taking $a = 0$ for the nominal case and applying the result of Theorem 2, we obtain, for a delay bound of $h = 10sec$, a near minimum attenuation level of $\gamma = 9.97$ for $\epsilon_f = 1e{-}7$. The filter matrix parameters for the nominal system are:

$$A_c = \begin{bmatrix} 0.2441 & -1.2208 \\ 0.4482 & -0.5902 \end{bmatrix},\ B_c = \begin{bmatrix} -12.83 & -3.99 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0.0621 & -0.1512 \\ 0 & 0 \end{bmatrix}.$$

Considering the uncertain case where $a \in [-0.06\ \ 0.06]$, and applying the various solution methods, the results of Table 1 are obtained. There, the quadratic H_∞ solution method was calculated by assigning a single Lyapunov function over the uncertainty polytope and is given in Theorem 2.

Solution Method	Stochastic H_∞ - γ
Nominal,	9.97, (Th. 1))
Quadratic	18.05, (Extended Th. 1)
RVD	15.3, $\bar{\epsilon} = 0.001,\ \epsilon_f = 1e-6$,[Th. 2]

Table 1: The results of the various solution methods used in the example. The robust quadratic solution was obtained by applying Theorem 1 to the uncertain system for $a \in [-0.06\quad 0.06]$ where a single set of Lyapunov function components (i.e. the decision variables) was applied. The "Robust Vertex-dependent" result refers to the application Theorem 2. RVD stands for Robust Vertex Dependent.

V. Conclusions

In this paper the theory of H_∞ estimation of state-multiplicative noisy Lipschitz-type nonlinear discrete-time retarded systems, is extended to the case where the system measurement is delayed rather than measured on-line. To the best of our knowledge, this problem has not been tackled before in the literature and certainly can not be simply derived from the linear counterpart solution.

We first solve the H_∞ estimation problem using a linear general-type filter for nominal systems and then we solve the robust case based on a specially devised Finsler lemma that allow us to extract a solution to the uncertain case via a tractable set of LMIs, with only two tuning parameters, a case which is not typical to other works in this field. We note that the latter solution is obtained via manipulation of the BRL for stochastic delayed system which, in turn, is based on an input-output approach. Some over-design is inherent to our solution due to the use of the bounded operators, used in the derivation of the latter BRL, which enable us to transform the retarded system to a norm-bounded one. Some additional over-design is also admitted in our solution due to the special structure imposed on R_2. The given numerical example brings the various solutions to the estimation process and demonstrates the tractability and applicability of our theory to control engineering scenarios, dealing with ever growing tasks of the modern era.

REFERENCES

[1] C. Y. Kao and B. Lincoln, "Simple stability criteria for systems with time-varying delays," *Automatica,* 40 (2004) 1429-1434.

[2] R. Wu, W. Zhang, F. Song, Z. Wu and W. Guo, "Observer-based stabilization of one-sided Lischitz systems with application to flexible link manipulator," *Advances in mechanical engineering,* 7(12) (2015) 1-8.

[3] A.M. Pertew, H.J. Marquez and Q. Zhao, "H_∞ observer design for Lipschitz nonlinear systems," *IEEE Trans. on Automat. Contr.,* 51(7) (2006) 1211-1216.

[4] P.R. Pagilla and Y. Zhu, "Controller and observer design for Lipschitz nonlinear systems," *Proceeding of the 2004 American Control Conference (ACC),* Boston, Massachusetts, USA, 2004.

[5] X. Yu and F. Liao, "Observer-based trajectory tracking control with preview action for a class of discrete-time Lipschitz nonlinear systems and its applications," *Advances in Mechanical Engineering,* 12(7) (2020) 1 -18.

[6] C.E. de Souza, "Robust H_∞ filtering for a class of discrete-time Lipschitz nonlinear systems," *Automtica,* 103 (2019) 69-80.

[7] W. Zhang, G. Feng and Q. Li, "Robust H_∞ filtering for general nonlinear stochastic state-delayed systems," *Mathematical problems in engineering*, doi:10.1155/2012/231352, 2012.

[8] W. Zhang, B.S. Chen, H. Tang, L. Sheng and M. Gao, "Some remarks on general nonlinear stochastic H_∞ control with state, control and disturbance-dependent noise," *IEEE Trans. on Automat. Contr.,* 59(1) (2014) 237-242.

[9] W. Zhang, G. Feng and Q. Li, "Nonlinear stochastic H_∞, H_2 control with (x-u-v)-dependant noise: Infinite horizon case," *IEEE Trans. on Automat. Contr.,* 53(5) (2008) 1323-1328.

[10] E. Gershon and U. Shaked, "Nonlinear state-multiplicative stochastic systems - Robust Vertex-dependant H_∞ Filtering," *Int. J. of Robust and Nonlinear Control*, under review.

[11] M. Pezzutto, L. Schenato and S. Dey, "Transmission scheduling for remote stimation with multi-packet reception under multisensor Interference," *IFAC PapersOnLine,* 53-2 (2020) 2628-2633.

[12] H. Fu, B. Cui, B. Zhuang and J. Zhang "State estimation for a class of distributed parameter systems with Time-Varying Delay over Mobile Sensor–Actuator networks with missing measurements," *Mathematics,* 9 (661) (2021) 1-18.

[13] K. Huang, W. Liu, M. Shirvanimoghaddam, Y. Li and V. Vucetic, "Real-time remote estimation with hybrid ARQ in wireless networked control," arXiv:1903.12472 cs.IT , 2021.

[14] E. Gershon, U. Shaked, I. Yaesh, "H_∞ control and Estimation of State-multiplicative linear systems," *Lecture Notes in Control and Information sciences, LNCIS, Springer,* 318 (2005).

[15] E. Gershon and U. Shaked, *Advanced topics in control and estimation of state-multiplicative noisy systems*, *Lecture Notes in Control and Information sciences, LNCIS, Springer,* 439 (2013).

[16] V. Dragan and T. Morozan, "Mixed input-output optimization for time-varying Ito systems with state dependent noise," *Dynamics of Continuous, Discrete and Impulsive Systems,* 3 (1997) 317-333.

[17] A. El, Bouhtouri, D. Hinriechsen and AJ. Pritchard, "H_∞-type control for discrete-time stochasic systems," *Int. J. Robust Nonlinear Control,* 9 (1999) 923-948.

[18] E. Gershon, U. Shaked, "H_∞ Output-Feedback of Discrete-time Systems with State-Multiplicative Noise," *Automatica,* 44(2) (2008) 574-579.

[19] P. Florchinger, E. I. Verriest, "Stabilization of Nonlinear Stochastic Systems with Dekay Feedback," *Proc. 32nd IEEE Conf. Decision and Control, San Antonio, TX,* pp. 859-860, December (1993).

[20] S. Xu, J. Lam, X. Mao and Y. Zou, "A new LMI condition for delay-dependent robust stability of stochastic time-delay systems," *Asian J. Control,* 7 (2005) 419-423.

[21] W. H. Chen, Z. Guan and X. Lu, "Delay-dependent exponential stability of uncertain stochastic systems with multiple delays: an LMI approach," *Systems and Control Letters,* 54 (2005) 547-555.

[22] E. I. Verriest, "Asymptotic Properties of Stochastic Delay Systems," *Advances in Time-Delay Systems, S-I. Niculesescu and K. Gu (eds.), Springer-Verlag,* (2004) 389-435.

[23] E. Gershon, U. Shaked, N. Berman, "H_∞ Control and Estimation of State-Multiplicative Stochastic Systems with Delay," *IEEE Trans. on Automat. Contr.,* 52(9) (2007) 329-334.

[24] E. Gershon, U. Shaked, "Robust H_∞ output-feedback control of state-multiplicative stochastic systems with delay," *International Journal of Robust and Nonlinear Control,* 21(11) (2011) 1283-1296.

[25] H. Gao, J. Lam and C. Wang,"Robust H_∞ filtering for discrete stochastic time-delay systems with nonlinear disturbances," *Nonlinear Dynamics and Systems Theory*, 4(3) (2004) 285 – 301.

[26] H. Gao and T. Chen, "New results on stability of discrete-time systems with time-varying state delay",*IEEE Trans. on Automat. Contr.,* 52(2) (2007) 328-334.

[27] D. Yue, E. Tian and Y. Zhang, "Stability analysis of discrete systems with stochastic delay and its applications," *International Journal of Innovative Computing, Information and Control*, 5(8) (2009) 2391 - 2403.

[28] E. Gershon and U. Shaked, *Advances in H_∞ control theory: Switched, Delayed and Biological Systems, Lecture Notes in Control and Information sciences, LNCIS, Springer,* 481 (2019).

[29] E. Gershon and U. Shaked, "Vertex-dependent Filtering of Retarded Uncertain State-multiplicative Noisy Systems" *International Journal of Robust and Nonlinear Control,* 30(16) (2020) 6978-6993.

[30] E. Gershon and U. Shaked, "Retarded State-multiplicative Stochastic Systems - Robust Vertex-dependent H_∞ and H_2 Filtering," *International Journal of Control,* 95(10), (2021) 2865–2875.