The Interacting Multiple Model Feedback Particle Filter for the State-dependent Switching Diffusion Systems

Yiyang Chen, Ruoyu Wang, Dengyu Yang and Xue Luo *senior member, IEEE*

***Abstract*—In this paper, we develop the interacting multiple model-feedback particle filter (IMM-FPF) for the state-dependent switching diffusion systems (SD-SDS). Starting from the Kushner's equation of the hybrid state, we derive the evolutionary equations for the modes' and the continuous states' probabilities given the mode. The IMM is used to merge the estimate of the continuous states from all the modes. As for the implementation, the constant gain approximation is used in the FPF. At last, the numerical example of a three-mode SD-SDS is experimented. Both the accuracy and the efficiency are compared with the IMM-particle filter.**

***Index Terms*—nonlinear filtering, switching diffusion process, interacting multiple model, feedback particle filter**

I. Introduction

Switching diffusion processes (SDS) are kinds of nonlinear, time-continuous stochastic hybrid processes, which have been widely used in fault-tolerant control system, multi-target tracking and specialized production systems. It is a special case of the stochastic hybrid systems (SHS) [6], which describes the continuous state evolving according to the dynamics in different modes. The modes in the SHS are switched due to a kernel function, while in the SDS the switching of the modes is ruled by a transition rate matrix, and the continuous state in each mode is described by the diffusion process.

The filtering problem is to estimate the the continuous state from the observation data, no matter which mode it is in. The typical framework of the filtering algorithm for the SDS systems consists of three components [13]:

1) A filtering algorithm for estimating the continuous-valued state given a mode;
2) An association algorithm for linking modes to signal dynamics;
3) A fusion process for integrating the results of 1) and 2).

As for the first issue, the filtering algorithms: the classical Kalman filter [7] and its derivatives may perform poorly for the problems with the high nonlinearity and the non-Gaussian initial distribution. In the 1990s, the particle filter (PF) is studied and yields fairly satisfactory results in general, see the tutorial [1]. However, the PF may suffer from the issues, such as particle degeneration. In 2013, Yang et al. [15] proposed the feedback particle filter (FPF), motivated by the mean field game theory, which combines the feedback structure and the flexibility of the particles. Compared with the PF, it doesn't need any resampling strategy.

As for the third issue, the fusion process: Multiple model (MM) filters [12] provide the exact solutions in the discrete time settings, but the number of filters increases exponentially over time. The merging process was exploited to reduce the number in the interacting multiple model (IMM) filter [3].

In the original IMM [3], the Kalman filter is used in each mode. It is effective in the linear SHS with continuous state-dependent mode transitions [5]. When dealing with the nonlinear systems, the IMM can't yield good estimate in general. Bar-Shalom et al. [4] proposed the interacting multiple model-particle filter (IMM-PF) to estimate the continuous state with constant transition rate matrix. Yang et al. [13] improved the filtering algorithm in IMM-PF by replacing PF with FPF, named IMM-FPF, and verified its effectiveness for nonlinear systems with constant transition rate matrix. Later, Liu et al. [8] derived the Kushner's equation for generalized SHS with the transition kernel. The Kushner's equation describes the evolution of the posterior distribution of the continuous state. Then, they proposed a finite difference-based filtering algorithm.

In this paper, we not only broaden the simplifying assumption of state-independent switching, but aslo develop the IMM-FPF instead of IMM-KF to the SDS with *state-dependent* transition rate matrix, which are different from [2]. Starting from the Kushner's equation for the hybrid states, we first derive both the evolutionary equations of the modes' probability $\pi_t(q)$ and the conditional continuous states' probabilities $\pi_t(z|q)$ in section III-A. The FPF is used to estimate z_t in each mode. Then we merge the processes by the IMM. The IMM-FPF algorithm is summarized in section III-C. In section IV, we numerically experiment the IMM-FPF for a three-mode SD-SDS, and compare it with the IMM-PF in both accuracy and efficiency. The conclusion arrives in the end.

II. Preliminary

A. The state-dependent switching diffusion systems (SD-SDS)

The SD-SDS is a stochastic process defined on the hybrid state space, which consists of finite many copies of the continuous state spaces. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete

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All the authors are with School of Mathematical Sciences, Beihang University, Beijing, 102206, P. R. China. X. Luo is also with the Key Laboratory of Mathematics, Informatics and Behavioral Semantics (LMIB), Beihang University, Beijing, 100191, P. R. China.

Y. Chen and R. Wang are co-first authors. X. Luo is the corresponding author. Email: xluo@buaa.edu.cn.

probability space. The SD-SDS is described by the following components:

- ◇ The hybrid state space $X := \mathbb{R}^d \times \mathscr{Q}$. The hybrid state is denoted as $x := (z, q) \in X$, where $z \in \mathbb{R}^d$ is the continuous state, and $q \in \mathscr{Q}$ is discrete, to symbolize the mode in the hybrid system.
- ◇ Continuous dynamics of $z \in \mathbb{R}^d$. For each discrete mode $q \in \mathscr{Q}$, the continuous state $z_t \in \mathbb{R}^d$ evolves according to a stochastic differential equation (SDE):

$$dz_t = a(z_t, q)dt + b(z_t, q)dw_t^q, \tag{2.1}$$

  where $a(z_t, q)$ is the drift term and $b(z_t, q) \neq 0$ is the diffusion term, w_t^q is the standard d-dimensional Brownian motion. As in [11], the Lipschitz condition with respect to z is imposed to $a(z, q)$ and $b(z, q)$ to ensure the existence and uniqueness of the solution to (2.1).
- ◇ The switching between the modes. The switching of the modes is ruled by a Markov chain:

$$\mathbb{P}(q_{t+\triangle t} = m | q_t = l) = \lambda_{lm}(x_t)\triangle t + o(\triangle t),$$

  if $m \neq l$, and $\lambda_{ll}(x_t) = \sum_{\substack{m=1\\m\neq l}}^{Q} \lambda_{lm}(x_t)$, where $Q = \sharp\mathscr{Q}$ is the number of the modes in $\mathscr{Q}$. The jumping process is generated by a state-dependent transition rate matrix, i.e.

$$\Lambda(x) = \begin{pmatrix} -\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1Q} \\ \lambda_{21} & -\lambda_{22} & \cdots & \lambda_{2Q} \\ \vdots & \vdots & & \vdots \\ \lambda_{Q1} & \lambda_{Q2} & \cdots & -\lambda_{QQ} \end{pmatrix}_{Q\times Q} (x).$$

It is well-known that the hybrid state in SD-SDS is a Markov process [6].

Besides the evolution of the hybrid state x_t given in section II-A, the observation is described by a continuous process $y_t \in \mathbb{R}^m$ satisfying

$$dy_t = h(x_t)dt + dB_t, \tag{2.2}$$

where B_t is a m-dimensional Brownian motion, independent of the Brownian motion w_t^q, $q \in \mathscr{Q}$, in (2.1).

The ultimate goal of the filtering problem is that given the observation history up to t, denoted as $\mathscr{F}_t^y := \sigma\{y_s : 0 \leq s \leq t\}$, for any $t \geq 0$, $\Gamma \in \mathscr{B}(X)$, one needs to obtain the distribution conditioned on $\mathscr{F}_t^y$, denoted as π_t, i.e.

$$\mathbb{P}\{x_t \in \Gamma | \mathscr{F}_t^y\} = \int_\Gamma \pi_t(x)dx,$$

or the expectation $\mathbb{E}x_t = \int_X x\pi_t(x)dx$.

B. Interacting multiple model-feedback particle filter (IMM-FPF) for the state-independent SDS

A typical filtering algorithm for the hybrid system is comprised of three parts: a filtering algorithm to estimate the state given the mode, i.e. $\pi_t(z|q)$, an association algorithm to associate the modes, i.e. $\pi_t(q)$ and a merging process to combine the above two, i.e. $\pi_t(x) = \pi_t(q)\pi_t(z|q)$. H. Blom [2] derived the evolutions of the densities:

$$\begin{aligned} d\pi_t(q) = &\sum_{l=1}^{Q} \Lambda_{lq}\pi_t(l)dt \\ &+ \left(\pi_t^{\cdot|q}[h] - \pi_t[h]\right)^T \pi_t(q)\left(dy_t - \pi_t[h]dt\right), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} d\pi_t(z|q) =& \mathcal{L}_c^*\pi_t(z|q)dt \\ &+ \pi_t(z|q)\left(h(x) - \pi_t^{\cdot|q}[h]\right)^T\left(dy_t - \pi_t^{\cdot|q}[h]dt\right) \\ &+ \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\Lambda_{lq}\pi_t(l)\left[\pi_t(z|l) - \pi_t(z|q)\right]dt, \end{aligned} \tag{2.4}$$

respectively, where

$$\pi_t[h] := \int_X \pi_t(x)h(x)dx, \tag{2.5}$$

$$\pi_t^{\cdot|q}[f] := \int_{\mathbb{R}^d} f(z, q)\pi_t(z|q)dz, \tag{2.6}$$

and

$$\begin{aligned} \mathcal{L}_c^*\pi_t(x) := &-\sum_{i=1}^{d}\frac{\partial}{\partial z_i}\left[a_i(x)\pi_t(x)\right] \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial z_i\partial z_j}\left[(bb^T)_{ij}(x)\pi_t(x)\right]. \end{aligned} \tag{2.7}$$

Here, we emphasize that Λ is a constant matrix.

The IMM-FPF for the state-independent SDS proposed in [13] essentially uses the FPF [15] to replace the Kalman filter to yield a more accurate estimate for $\pi_t(z|q)$ in the original IMM [3]. The FPF is a feedback control-based filter algorithm, where the particles empirically represent the continuous state.

Given the mode $q \in \mathscr{Q}$, the state of the ith particle evolves according to

$$\begin{aligned} dz_t^{i,q} =& a\left(z_t^{i,q}, q\right)dt + b\left(z_t^{i,q}, q\right)dw_t^{i,q} \\ &+ K^q\left(z_t^{i,q}, t\right)dI_t^q + u^q\left(z_t^{i,q}, z_t^{i,-q}, t\right)dt, \end{aligned} \tag{2.8}$$

$i = 1, \cdots, N_p^q$, where

$$dI_t^q := dy_t - \pi_t^{\cdot|q}[h]dt \tag{2.9}$$

is the innovation process, N_p^q is the number of particles in mode q, $z_t^{i,-q} = \{z_t^{i,l}\}_{l\neq q}$, $\{w_t^{i,q}\}_{i=1}^{N_p^q}$ are mutually independent standard Wiener processes, and K^q and u^q are the gain functions related to the continuous state in mode q and the interaction between the states in mode q and all the other modes, respectively.

It is shown in Theorem 2, [13] that if

$$\nabla\cdot\left(\hat{\pi}_t(z|q)K^q(z, t)\right) = -\hat{\pi}_t(z|q)\left(h(x) - \hat{\pi}_t^{\cdot|q}[h]\right)^T, \tag{2.10}$$

and

$$\nabla\cdot(\hat{\pi}_t(z|q)u^q(z,t)) = \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\Lambda_{lq}\pi_t(l)\left[\hat{\pi}_t(z|q)-\hat{\pi}_t(z|l)\right]$$
$$+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial z_i\partial z_j}\left(\hat{\pi}_t(z|q)\left(K^qK^{qT}\right)_{ij}(z,t)\right), \qquad (2.11)$$

respectively, where $\nabla\cdot$ is the divergence operator, $\hat{\pi}_t(z|q)$ is the empirical density of the particles $\{z_t^{i,q}\}_{i=1}^{N_p^q}$, then for all $q\in\mathcal{Q}$, given the same initial distribution $\pi_0(z|q)$, one has

$$\hat{\pi}_t(z|q)=\pi_t(z|q),$$

for all $t\geq 0$, where $\pi_t(z|q)$ is the solution to (2.4).

The last step is to merge the empirical densities of the state in each mode, i.e.

$$\pi_t(z)\approx\sum_{q=1}^{Q}\pi_t(q)\hat{\pi}_t(z|q). \qquad (2.12)$$

Consequently, the expectation of z_t is approximated by

$$\mathbb{E}z_t\approx\sum_{q=1}^{Q}\pi_t(q)\frac{1}{N_p^q}\sum_{i=1}^{N_p^q}z_t^{i,q}. \qquad (2.13)$$

III. Derivation of IMM-FPF for the SD-SDS

As described in section II-B, the densities $\pi_t(q)$ and $\pi_t(z|q)$ are synchronized by the law of total probability (2.12). Thus, we shall derive the evolution equations of $\pi_t(q)$ and $\pi_t(z|q)$ in the *state-dependent* SDS, similar as (2.3)-(2.4) in the state-independent case. Consequently, the corresponding IMM-FPF for the SD-SDS is proposed in section III-B.

A. The evolution SDEs of $\pi_t(q)$ and $\pi_t(z|q)$

It is obtained in section 2.B, [2] that the distribution of the hybrid state conditioned on the observation history $\pi_t(x)$ satisfies a stochastic partial differential equation (SPDE), similar as the Kushner's equation in the literature of the classical nonlinear filtering.

Theorem 1. *The evolution of $\pi_t(x)=\pi_t(z,q)$ satisfies*

$$d\pi_t(x)=\mathcal{L}_c^*\pi_t(x)dt+\mathcal{L}_d^*\pi_t(x)dt$$
$$+\left(h(x)-\pi_t[h]\right)^T\pi_t(x)\left(dy_t-\pi_t[h]dt\right), \qquad (3.14)$$

where

$$\mathcal{L}_d^*\pi_t(x):=\sum_{l=1}^{Q}\Lambda_{lq}(z,l)\pi_t(z,l). \qquad (3.15)$$

From the joint distribution $\pi_t(x)$ in (3.14), it is not hard to derive the evolution equations for $\pi_t(q)=\int_{\mathbb{R}^d}\pi_t(x)dz$ and $\pi_t(z|q)=\dfrac{\pi_t(x)}{\pi_t(q)}$.

Proposition 1. *The probability density of each mode $q\in\mathcal{Q}$ satisfies the SDE*

$$d\pi_t(q)=\sum_{l=1}^{Q}\pi_t^{\cdot|l}[\Lambda_{lq}]\pi_t(l)dt \qquad (3.16)$$
$$+\left(\pi_t^{\cdot|q}[h]-\pi_t[h]\right)^T\pi_t(q)\left(dy_t-\pi_t[h]dt\right),$$

where $\pi_t^{\cdot|q}[h]$ and $\pi_t[h]$ are in (2.5)-(2.6)*, respectively.*

Proof. It is clear that (3.16) is obtained by integrating (3.14) with respect to z, i.e.

$$d\pi_t(q)=\left(\int_{\mathbb{R}^d}\sum_{l=1}^{Q}\Lambda_{lq}(z,l)\pi_t(z,l)dz\right)dt$$
$$+\left(\int_{\mathbb{R}^d}\left(h(x)-\pi_t[h]\right)^T\pi_t(z|q)dz\right)\pi_t(q)\left(dy_t-\pi_t[h]dt\right)$$

by (2.7) and (3.15), since $\int_{\mathbb{R}^d}\mathcal{L}_c^*\pi_t(z,q)dz=0$ by (2.7). □

To obtain the SDE of $\pi_t(z|q)$, one needs to use the Itô formula.

Proposition 2. *Given the mode $q\in\mathcal{Q}$, the probability density of the continuous state $z\in\mathbb{R}^d$ conditioned on the mode q satisfies the SDE*

$$d\pi_t(z|q) \qquad (3.17)$$
$$=\mathcal{L}_c^*\pi_t(z|q)dt+\pi_t(z|q)\left(h(x)-\pi_t^{\cdot|q}[h]\right)^T\left(dy_t-\pi_t^{\cdot|q}[h]dt\right)$$
$$+\frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t(l)\left[\Lambda_{lq}(z,l)\pi_t(z|l)-\pi_t^{\cdot|l}[\Lambda_{lq}]\pi_t(z|q)\right]dt,$$

where $\pi_t^{\cdot|q}[h]$ and $\pi_t[h]$ are in (2.5)-(2.6)*, respectively.*

Proof. By the Itô's formula, one has

$$d\pi_t(z|q)=d\left(\frac{\pi_t(x)}{\pi_t(q)}\right) \qquad (3.18)$$
$$=\pi_t(x)d\left(\frac{1}{\pi_t(q)}\right)+\frac{d\pi_t(x)}{\pi_t(q)}+d\pi_t(x)\cdot d\left(\frac{1}{\pi_t(q)}\right).$$

Thus, we use the Itô's formula to give the SDE for $\dfrac{1}{\pi_t(q)}$:

$$d\left(\frac{1}{\pi_t(q)}\right)=-\frac{1}{\pi_t^2(q)}d\pi_t(q)+\frac{1}{\pi_t^3(q)}(d\pi_t(q))^2$$
$$\overset{(3.16)}{=}\frac{1}{\pi_t^2(q)}\left\{-\sum_{l=1}^{Q}\pi_t^{\cdot|l}[\Lambda_{lq}]\pi_t(l)+\pi_t(q)\left|\pi_t^{\cdot|q}[h]-\pi_t[h]\right|^2\right\}dt$$
$$-\frac{1}{\pi_t(q)}\left[\pi_t^{\cdot|q}[h]-\pi_t[h]\right]^T\left(dy_t-\pi_t[h]dt\right), \qquad (3.19)$$

where $|\circ|$ is the Euclidean norm in $\mathbb{R}^d$. Substituting (3.14) and (3.19) back to (3.18), by direct computations, one has

$$d\pi_t(z|q)=\pi_t(z|q)\left(\pi_t^{\cdot|q}[h]-h(x)\right)^T\left(\pi_t^{\cdot|q}[h]-\pi_t[h]\right)dt$$
$$+\frac{1}{\pi_t(q)}\mathcal{L}_c^*\pi_t(x)dt+\frac{1}{\pi_t(q)}\mathcal{L}_d^*\pi_t(x)dt$$
$$-\frac{\pi_t(z|q)}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t^{\cdot|l}[\Lambda_{lq}]\pi_t(l)dt \qquad (3.20)$$
$$+\pi_t(z|q)\left(h(x)-\pi_t^{\cdot|q}[h]\right)^T\left(dy_t-\pi_t[h]dt\right).$$

Recall the definition of $\mathcal{L}_c^*$ and $\mathcal{L}_d^*$ in (2.7) and (3.15), the second and third terms on the right-hand side of (3.20) become

$$\frac{1}{\pi_t(q)}\mathcal{L}_c^*\pi_t(x)\overset{(2.7)}{=}\mathcal{L}_c^*\pi_t(z|q), \qquad (3.21)$$

and

$$\frac{1}{\pi_t(q)}\mathcal{L}_d^*\pi_t(x) \overset{(3.15)}{=} \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\Lambda_{lq}(z,l)\pi_t(z|l)\pi_t(l), \quad (3.22)$$

respectively. Therefore, (3.17) is obtained by substituting (3.21)-(3.22) into (3.20). □

B. The IMM-FPF for the SD-SDS

Similar as in section II-B, the IMM-FPF for the SD-SDS is comprised of three steps:

(1) The FPF to estimate the state given the mode, $\pi_t(z|q)$;

(2) The association algorithm to approximate the probability of the mode $\pi_t(q)$;

(3) The merging process to combine the above two by (2.12).

Step (3) are straight-forward after obtaining $\pi_t(q)$ and $\pi_t(z|q)$. Step (2) is to solve (3.16) numerically. Compared with (2.3), the only difference is the term containing Λ. If Λ is state-independent, (3.16) simply reduces to (2.3), since

$$\pi_t^{\cdot|l}[\Lambda_{lq}] = \Lambda_{lq}\int_{\mathbb{R}^d}\pi_t^{\cdot|l}(z)dz = \Lambda_{lq}. \quad (3.23)$$

Step (1) requires some modifications of the FPF in section II-B. We shall detail them from the viewpoint of splitting-up scheme [9] in the sequel.

Given the mode $q \in \mathscr{Q}$, we evolve N_p^q particles to empirically approximate $\pi_t(z|q)$. The ith particle evolves according to (2.8). Thus, the corresponding forward Kolmogorov equation is

$$\begin{aligned} d\hat{\pi}_t(z|q) =& \mathcal{L}_c^*\hat{\pi}_t(z|q)dt \quad (3.24)\\ &- \nabla\cdot(\hat{\pi}_t(z|q)K^q(z,t))\,dI_t^q - \nabla\cdot(\hat{\pi}_t(z|q)u^q(z,t))\,dt \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial z_i\partial z_j}\left(\hat{\pi}_t(z|q)\left(K^q{K^q}^T\right)_{ij}(z,t)\right)dt, \end{aligned}$$

where dI_t^q is the innovation process (2.9). By equating the terms in front of dI_t^q and dt in (3.17) and (3.24), we have the same equation (2.10) for K^q, but the equation for u^q is different in this state-dependent case, i.e.

$$\begin{aligned} &\nabla\cdot(\hat{\pi}_t(z|q)u^q(z,t)) \quad (3.25)\\ =& -\frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t(l)\left[\Lambda_{lq}(z,l)\hat{\pi}_t(z|l) - \pi_t^{\cdot|l}[\Lambda_{lq}]\hat{\pi}_t(z|q)\right] \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial z_i\partial z_j}\left[\hat{\pi}_t(z|q)\left(K^q{K^q}^T\right)_{ij}(z,t)\right]. \end{aligned}$$

Again if Λ is state-independent, (3.25) simply reduces to (2.11) due to (3.23).

C. Algorithm

In this subsection, we shall discuss the implementation of IMM-FPF for the SD-SDS. The main difficulty is to obtain an approximate solution to (2.10) and (3.25). The *constant gain approximation* of K^q and u^q are proposed and justified in [14] and applied in [13] in the state-independent SDS. Since (2.10) is the same as that in [13], the constant gain K^q is still

$$K^q \approx \frac{1}{N_p^q}\sum_{i=1}^{N_p^q} z_t^{i,q}\left(h(z_t^{i,q},q) - \hat{\pi}_t^{\cdot|q}[h]\right)^T, \quad (3.26)$$

where

$$\hat{\pi}_t^{\cdot|q}[h] \approx \frac{1}{N_p^q}\sum_{j=1}^{N_p^q}h(z_t^{j,q},q). \quad (3.27)$$

To approximate u^q, we claim that

$$\begin{aligned} &\nabla\cdot\left\{\hat{\pi}_t(z|q)\left(-\frac{1}{2}K^q(z,t)\left(h(x)-\hat{\pi}_t^{\cdot|q}[h]\right)+\Omega(z,t)\right)\right\} \\ =& \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial z_i\partial z_j}\left[\hat{\pi}_t(z|q)\left(K^q{K^q}^T\right)_{ij}(z,t)\right], \quad (3.28) \end{aligned}$$

where $\Omega_l = \frac{1}{2}\sum_{k,s=1}^{d}K_{ks}^q\frac{\partial K_{ls}^q}{\partial z_k}$. Due to the page limit, we only show (3.28) for $d=1$.

$$\begin{aligned} &\left\{\hat{\pi}_t(z|q)\left(-\frac{1}{2}K^q(z,t)\left(h(x)-\hat{\pi}_t^{\cdot|q}[h]\right)+\frac{1}{2}K^q{K^q}'(z,t)\right)\right\}' \\ \overset{(2.10)}{=}& \left\{\hat{\pi}_t(z|q)\frac{1}{\hat{\pi}_t(z|q)}\frac{1}{2}K^q(z,t)\left(\hat{\pi}_t(z|q)K^q(z,t)\right)' \right.\\ &\left.+\frac{1}{2}\hat{\pi}_t(z|q)K^q{K^q}'(z,t)\right\}' \end{aligned}$$

Comparing the right-hand side of (3.28) with the last term on the right-hand side of (3.25), we find part of u^q. Let us denote it as

$$\begin{aligned} u_1^q(z,t) =& -\frac{1}{2}K^q(z,t)\left(h(x)-\hat{\pi}_t^{\cdot|q}[h]\right)+\Omega(z,t) \\ \overset{(3.26)}{\approx}& -\frac{1}{2}K^q\left(h(x)-\hat{\pi}_t^{\cdot|q}[h]\right) \\ \approx& \frac{1}{2}K^q\left(h(z,q)-\frac{1}{N_p^q}\sum_{j=1}^{N_p^q}h(z_t^{j,q},q)\right), \quad (3.29) \end{aligned}$$

since K^q is approximated by a constant. Then, we assume the remaining part of u^q, denoted as $u_2^q := u^q - u_1^q$, is also a constant with respect to z. Multiplying z and integrating in $\mathbb{R}^d$, we have

$$\begin{aligned} u_2^q \approx& -\int_{\mathbb{R}^d} z\nabla\cdot(\hat{\pi}_t(z|q)u_2^q)\,dz \quad (3.30)\\ =& \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t(l)\int_{\mathbb{R}^d} z\left[\Lambda_{lq}(z,l)\hat{\pi}_t(z|l) - \hat{\pi}_t^{\cdot|l}[\Lambda_{lq}]\hat{\pi}_t(z|q)\right]dz \\ =& \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t(l)\left[\hat{\pi}_t^{\cdot|l}[z\Lambda_{lq}(z,l)] - \hat{\pi}_t^{\cdot|l}[\Lambda_{lq}]\hat{\pi}_t^{\cdot|q}[z]\right] \\ \approx& \frac{1}{\pi_t(q)}\sum_{l=1}^{Q}\pi_t(l)\left\{\frac{1}{N_p^l}\sum_{i=1}^{N_p^l}\Lambda_{lq}(z_t^{i,l},l)\left(z_t^{i,l}-\frac{1}{N_p^q}\sum_{j=1}^{N_p^q}z_t^{j,q}\right)\right\}, \end{aligned}$$

where $\hat{\pi}_t^{\cdot|l}[z\Lambda_{lq}(z,l)] \approx \frac{1}{N_p^l}\sum_{i=1}^{N_p^l}z_t^{i,l}\Lambda_{lq}(z_t^{i,l},l)$, $\hat{\pi}_t^{\cdot|l}[\Lambda_{lq}] \approx \frac{1}{N_p^l}\sum_{i=1}^{N_p^l}\Lambda_{lq}(z_t^{i,l},l)$ and $\hat{\pi}_t^{\cdot|q}[z] \approx \frac{1}{N_p^q}\sum_{i=1}^{N_p^q}z_t^{i,l}$. Therefore, one obtains $u^q(z,t) = u_1^q(z,t) + u_2^q$ by (3.29) and (3.30).

We summarize the implementation of the IMM-FPF of the SD-SDS in Algorithm 1.

Algorithm 1 The IMM-FPF for the SD-SDS

% Initialization
for $q \in \mathscr{Q}$ **do**
  $\pi_0(q) = \frac{1}{Q}$
  for $i = 1$ to N_p^q **do**
    Sample $z_0^{i,q}$ from $\pi_0(z|q)$
  end for
end for
for $t = 0$ to $T - \Delta t$ with time step Δt **do**
  for $q \in \mathscr{Q}$ **do**
    Calculate $\hat{\pi}_t^{\cdot|q}[h]$ approximately by (3.27);
  end for
  % the FPF for each mode $q \in \mathscr{Q}$
  Calculate $\pi_t[h] \approx \sum_{q \in \mathscr{Q}} \pi_t(q) \hat{\pi}_t^{\cdot|q}[h]$;
  for $q \in \mathscr{Q}$ **do**
    Generate N_p^q independent samples $\Delta w_t^{i,q}$ from $\mathcal{N}(0, I_{d\times d}\Delta t)$, where $I_{d\times d}$ is an identity matrix;
    Calculate (K^q, u^q) for each particle $z_t^{i,q}$, $i = 1, \cdots, N_p^q$, by (3.26) and (3.29)-(3.30), respectively;
    Evolve the particles $\{z_t^{i,q}\}_{i=1}^{N_p^q}$ according to (2.8)

$$z_{t+\Delta t}^{i,q} = z_t^{i,q} + a(z_t^{i,q}, q)\Delta t + b(x_t^{i,q}, q)\Delta w_t^{i,q} + K^q \Delta I_t^q + u^q(z_t^{i,q}, t)\Delta t,$$

    where $\Delta I_t^q = \Delta y_t - \hat{\pi}_t^{\cdot|q}[h]\Delta t$;
    Evolve $\pi_t(q)$ by (3.16) approximately:

$$\pi_{t+\Delta t}(q) \approx \pi_t(q) + \sum_{l=1}^{Q} \hat{\pi}_t^{\cdot|l}[\Lambda_{lq}]\pi_t(l)\Delta t + \left(\hat{\pi}_t^{\cdot|q}[h] - \pi_t[h]\right)^T \pi_t(q)\left(\Delta y_t - \pi_t[h]\Delta t\right),$$

    where $\hat{\pi}_t^{\cdot|l}[\Lambda_{lq}]$ is approximated similarly as that in (3.27).
  end for
  The continuous state $z_{t+\Delta t}$ is approximated by (2.13):

$$\hat{z}_{t+\Delta t} \approx \sum_{q\in\mathscr{Q}} \pi_{t+\Delta t}(q) \frac{1}{N_p^q} \sum_{i=1}^{N_p^q} z_{t+\Delta t}^{i,q};$$

end for

IV. Numeric

In this section, we shall verify the feasibility of the IMM-FPF for the SD-SDS numerically. It is shown that the IMM-FPF outperforms the interacting multiple mode-particle filter (IMM-PF) in both accuracy and efficiency.

An example of a 3-mode scalar-state with Markov switching process is considered below. The transition rate matrix is state-dependent. We detail the setting-up of the SD-SDS as follows:

- ◇ There are total three modes in $\mathscr{Q}$, labeled as $\{1, 2, 3\}$. The corresponding state equation in each mode is given

$$\begin{cases} q = 1: & dz_t = z_t\left(1 - z_t^2\right)dt + \sigma_w dw_t \\ q = 2: & dz_t = \arcsin\left(0.05 z_t\right)dt + \sigma_w dw_t \\ q = 3: & dz_t = 3\sin z_t dt + \sigma_w dw_t \end{cases}. \tag{4.31}$$

- ◇ The modes evolve as a right-continuous Markov process with state-dependent transition rate matrix

$$\Lambda(z_t) = \begin{pmatrix} -0.1 & 0.1 & 0 \\ 0.05 & -0.1 & 0.05 \\ 0 & 0.1 & -0.1 \end{pmatrix}\left(1 + \sin z_t^2\right) \tag{4.32}$$

- ◇ The observation process is given by

$$dy_t = z_t dt + dv_t, \tag{4.33}$$

  which is a linear observation.

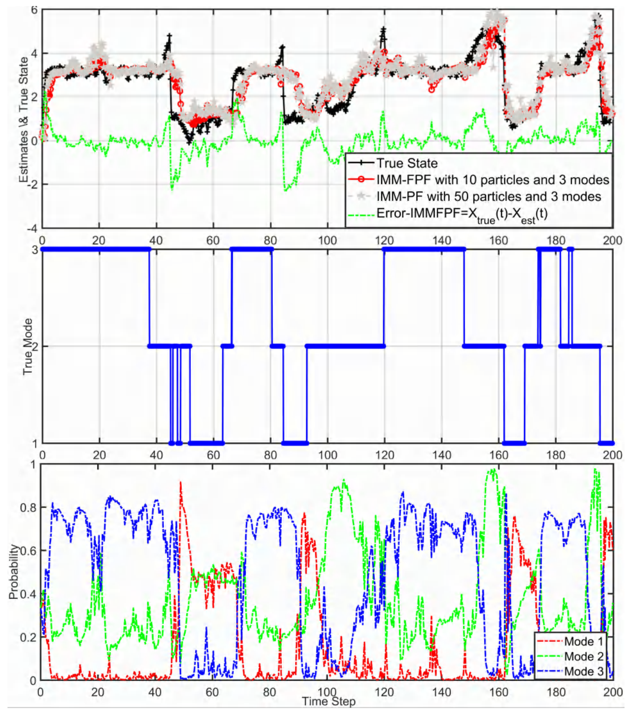

Fig. 1. The figure at the top is the estimates of z_t with the different algorithms. Red: Estimate obtained by IMM-FPF with 50 particles. Grey: Estimate obtained by IMM-PF with 50 particles. Black: True state. Green: Error between the IMM-FPF and the true state. The figure in the middle: True mode transition. The figure at the bottom: the estimated probabilities of three modes, i.e. $\pi_t(q)$, $q = 1, 2, 3$.

In this numerical experiment, we set the total time $T = 200$, with time step $\Delta t = 0.005$. The covariance of dw_t and dv_t are both 1, $\sigma_w = 0.1$. The initial $N_p^q = 10$ particles are sampled from $\mathcal{N}(0.1, 1)$ for each mode $q = 1, 2, 3$, and the initial mode weight is assumed to be $\pi_0(1) = \pi_0(2) = \pi_0(3) = \frac{1}{3}$. The true initial mode is randomly generated based on initial mode weight. The true state z_t and the mode weight $\pi_t(q)$ are generated by the Euler-Maruyama method, according to (4.31) and (4.32), respectively.

We compared the accuracy and efficiency of IMM-FPF with the IMM-PF for the SD-SDS (4.31)-(4.33). We also tried the classical IMM filter [3] several times for this system without even one non-explosion realization. It may due to the nonlinearity in each mode, see (4.31). Thus, we don't include the IMM's results in this paper. The IMM-PF is an improved PF for Markov jump nonlinear systems using the idea of Rao-Blackwellization [10]. It uses PF in parallel in each mode, and finally estimates the state by weighted average of state estimation for each mode. In the IMM-PF, we set the number of the particles to be $N_p^q = 50$ in each mode. The particles are resampled at each time step. Other parameters are the same in both IMM-FPF and IMM-PF.

Fig. 1 shows the performance of IMM-FPF of the SD-SDS (4.31)-(4.33). From the figure at the top, We can see when the state changes dramatically when the true mode transits, both the IMM-FPF and the IMM-PF can react immediately and show good tracking ability. To measure the algorithm's error, we use the mean square error (MSE), which is defined for one realization as

$$\left[\sum_{t_i=0}^{T}(z_{real}(t_i) - z_{est}(t_i))^2\right]^{\frac{1}{2}},$$

where $z_{real}(t_i)$ and $z_{est}(t_i)$ are the true state and the estimation at time t_i, respectively. The CPU time for the IMM-PF with 50 particles are 5.6s, and the MSE for this realization is 158.39. On the contrary, the CPU time of the IMM-FPF with 10 particles is 5.4s with the MSE being 118.42. It can be seen that with comparative CPU time, the IMM-FPF can achieve better accuracy than IMM-PF by using less number of particles. It means that taking advantage of the feedback mechanism, the IMM-FPF beats the IMM-PF in less storage space and shorter CPU running time.

The figure in the middle illustrates the true transition of the three modes, while that at the bottom is the corresponding probabilities of the continuous state z_t at time t in the three modes. It shows clearly that the modes' transition is described accurately by the probabilities. For example, when the true mode is mode 3 during time period around $0 \sim 40$, $70 \sim 80$, $120 \sim 150$, etc, the probability of mode 3 is obviously higher than the other two.

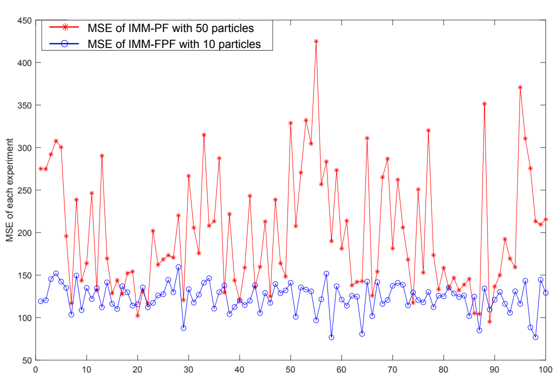

Fig. 2. Averaged MSE of 100 Monte Carlo simulations of the IMM-FPF.

We Monte Carlo 100 realizations of the experiment above. Fig. 2 shows that the averaged MSE over 100 realizations of IMM-FPF is 128.6, while that of IMM-PF is 180.1. It shows that only 7 out of 100 realizations, the IMM-PF has slightly smaller MSE than the IMM-FPF. The largest MSE of IMM-PF can reach almost 430, while that of IMM-FPF is only around 150. The smallest MSEs of both algorithms are around 100. Thus, the MSE of the IMM-FPF has smaller variance. It turns out that the IMM-FPF is more robust to the randomness of the particles and more stable than the IMM-PF.

V. Conclusion

In this paper, we develop the IMM-FPF for the SD-SDS. Within the typical framework of the filtering for the SDS, the IMM is used to merge the processes, and the FPF to estimate the continuous state in each mode. The key ingredient is to obtain the evolutionary equation for the modes' probabilities and the continuous states' probabilities given the mode in this state-dependent case. These are derived from the Kushner's equation of the hybrid state of the SD-SDS. As for the implementation of IMM-FPF, we use the constant gain approximation in the FPF, proposed in [13], but adapting to the state-dependent transition rate matrix. To verify the feasibility of IMM-FPF for the SD-SDS, we take a state-dependent one-dimensional SDS as an example, the accuracy and efficiency of the IMM-FPF to track the continuous state and to identify the mode at every time instant. The IMM-FPF outperforms the IMM-PF in accuracy and yields estimates with much smaller variance in MSE.

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